

FO Model Checking on Map Graphs

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Theorem (Main Result)

The model checking problem for first-order logic on the class of all map graphs is fixed-parameter tractable when parameterised by the size of the formula.

- G is a clique $\Leftrightarrow G \models \forall x \forall y (x = y \vee Exy)$
- G contains a k -clique \Leftrightarrow
 $G \models \exists x_1 \dots \exists x_k \bigwedge_{i < j} (\neg(x_i = x_j) \wedge Ex_i x_j)$
- G has dominating set of size $\leq k$ \Leftrightarrow
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- G is 3-colourable \Leftrightarrow

$$G \models \exists X \exists Y \exists Z \forall x \forall y ((Xx \vee Yx \vee Zx) \wedge$$
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- model checking: decide algorithmically whether $A \models \varphi$
- in general: A could be graph, database, ...

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for some $f : \mathbb{N} \rightarrow \mathbb{N}$ and $c \in \mathbb{N}$
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- brute force for FO gives

$$O(|V|^{\text{qr}(\varphi)}),$$

which is **not** *fpt*

- for MSO, even checking $G \models \varphi_{3\text{-col}}$ is NP-complete
(\leadsto *fpt* on arbitrary graphs unlikely)

Algorithmic Meta-Theorems

- Courcelle '90: MSO model checking is fixed-parameter tractable (even linear time) on [graphs] of bounded tree-width
- gives unified approach to various algorithmic problems, therefore called *algorithmic meta-theorem*

Algorithmic Meta-Theorems

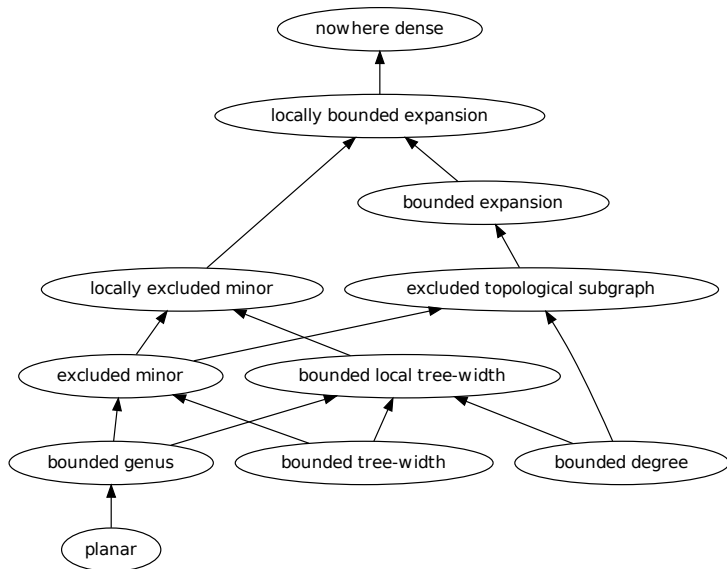
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- gives unified approach to various algorithmic problems, therefore called *algorithmic meta-theorem*
- power tool for FO: Gaifman Locality Theorem reduces to φ of the form

$$\exists x_1 \dots \exists x_k \left(\bigwedge_i \psi^{(r)}(x_i) \wedge \bigwedge_{i < j} d(x_i, x_j) > 2r \right)$$

for some $r \in \mathbb{N}$ and r -local formula $\psi^{(r)}$.

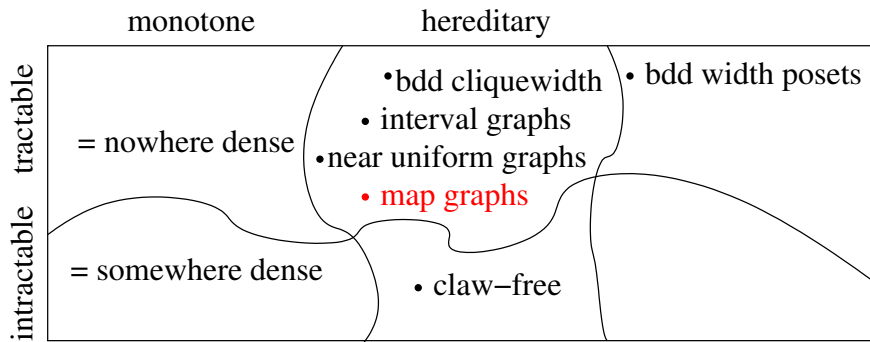
- reduces FO-model checking to:
 - evaluating $\psi^{(r)}$ on local parts of the graph and
 - finding $2r$ -independent subsets of a set $S \subseteq V$.

FO Model Checking on Monotone Graph Classes



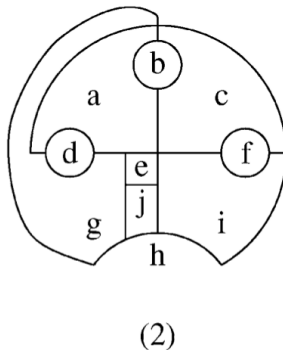
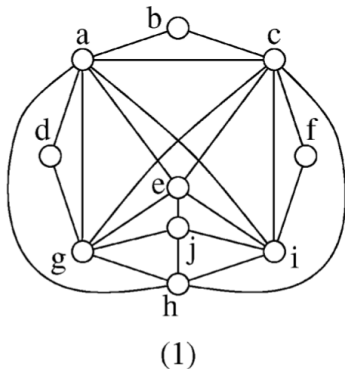
Known Results beyond Nowhere Dense

- Courcelle et al. (2000): bounded clique-width (even for MSO)
- Ganian et al. (2013): (certain) interval graphs, with given interval representation
- Gajarský et al. (2015): posets of bounded width
- Gajarský et al. (2016): near uniform graph classes



Map Graphs

- graph $G = (V, E)$ drawn in the plane with *disc homeomorph* D_v for each $v \in V$
- $uv \in E \iff D_u \cap D_v \neq \emptyset$



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- $uv \in E \iff D_u \cap D_v \neq \emptyset$
- planar graphs are map graphs
- may contain arbitrarily large cliques
- hereditary graph class (closed under induced subgraphs)
- not monotone (not closed under edge removal)
- in particular not minor closed, not characterised by excluded minors

Theorem (Main Result)

The model checking problem for first-order logic on the class of all map graphs is fixed-parameter tractable when parameterised by the size of the formula.

- does not generalise to relational structures whose Gaifman graph is a map graph
- does not generalise to edge-coloured map graphs

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- does not generalise to edge-coloured map graphs
- proof sketch: convert map graph G to nowhere dense graph G' such that G is FO-interpretable in G' :

$$\begin{aligned}v \in V(G) &\Leftrightarrow G' \models \nu(v) \\ uv \in E(G) &\Leftrightarrow G' \models \mu(u, v)\end{aligned}$$

with ν, μ in dependent of G .

Characterisation of map graphs

- map graphs first studied by Chen/Grigni/Papadimitriou (STOC1998)
- combinatorial characterisation:

Theorem

A graph $G = (V, E)$ is a map graph if, and only if, there is planar bipartite graph $H = (V \cup P, F)$ such that for all $u, v \in V$:

$$uv \in E \iff up, vp \in F \text{ for some } p \in P.$$

Moreover, we may choose P such that $|P| \leq 3|V| - 6$.

- H is called *witness* for G
- this puts recognising map graphs into NP

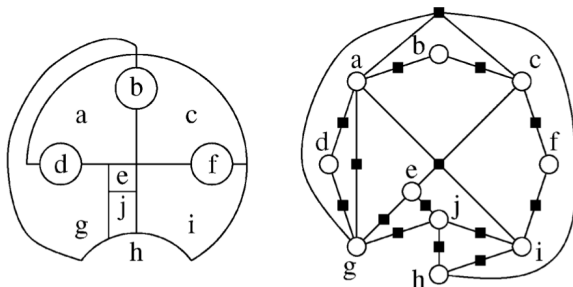
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Thorup's Algorithm

- given a witness graph, FO model checking is easy
- obvious question: can we find such an H efficiently?
- Mikkel Thorup (FOCS 1998): Map Graphs in Polynomial Time
- however...
 - recognition algorithm is only sketched
 - intricate case distinction, “remaining cases are similar”
 - journal version never appeared
 - not clear if it produces a witness graph
 - running time roughly $|V|^{120}$.
- recent work by Mnich, Rutter, Schmidt (SWAT 2016): Linear time algorithm, produces witness, but only if witness is outerplanar

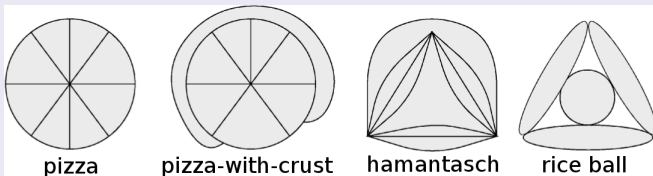
Our Algorithm

- basic idea: create bipartite graph $(V \cup S, F)$, one vertex $s_C \in S$ for each maximal clique $C \subseteq V$, connected to all $v \in C$.
- map graph can have at most $27|V|$ many maximal cliques, these can be enumerated in polynomial time
- however, the resulting graphs are not nowhere dense
- we use Chen et al.'s analysis of the possible types of cliques in map graphs

Cliques in Map Graphs

Theorem (Chen et al.)

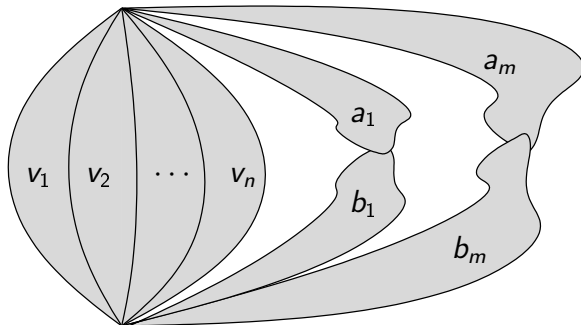
Every clique C in a map graph G is of one of the following four types:



Theorem (Chen et al.)

In a map graph $G = (V, E)$ there can be at most $27 \cdot |V|$ maximal cliques.

- large hamantasch-cliques lead to problems:



- in 3-connected map graphs, the vertices in each hamantasch have at most 11 different neighbourhood-types
- remove similar vertices from hamantaschen, encode multiplicities as colours of the remaining vertices
- use decomposition into 3-connected components for general map graphs (somewhat technical)

- now may assume: all large cliques are pizza or pizza-with-crust
- introduce maximum clique vertices s_C in descending order of size of C
- removing vertices from large cliques if they are already connected
- now we can show: resulting reduced max clique graph M is nowhere dense

Showing Nowhere Denseness

- graph class \mathcal{C} is nowhere dense iff for every $r \geq 1$ no $G \in \mathcal{C}$ contains an r -subdivision of $K_{\mathcal{C}(r)}$.
- let G be map graph, M reduced max clique graph, H some witness for G
- show: if M contains r -subdivision of $K_{\mathcal{C}}$, then H is not planar
- replace clique-vertices in M with vertices $p \in P$ in H
- may use $p \in P$ more than once, but bounded number of times
- may loose (lots of) edges in the process

Conclusion

- We have shown that FO model checking is fixed-parameter tractable on map graphs
- what about other hereditary graph classes? subclasses of $K_{1,3}$ -free graphs?
- how does this generalise to relational structures?
- how can graphs be reduced to nowhere dense graphs in general? (cf. Gajarský et al.)