

Decision Problems for Subclasses of Rational Relations over Finite and Infinite Words

Christof Löding¹ Christopher Spinrath²

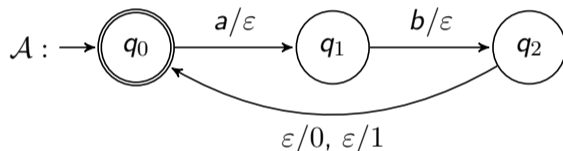
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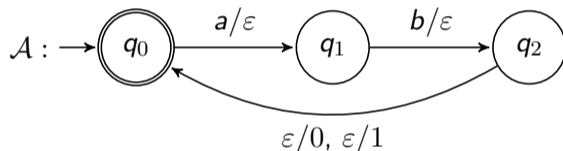
Rational Relations

- relations over **finite** words
- defined by **transducers** (**non-deterministic multi-tape** “automata”)
- **Example:**



Rational Relations

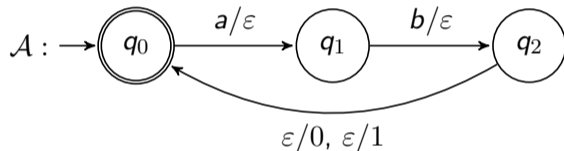
- relations over **finite** words
- defined by **transducers** (**non-deterministic multi-tape** “automata”)
- **Example:**



- $R_*(\mathcal{A}) = \{((ab)^n, v) \mid n \in \mathbb{N}, v \in \{0, 1\}^n\}$
- e.g. $(abab, 10), (abab, 00) \in R_*(\mathcal{A})$

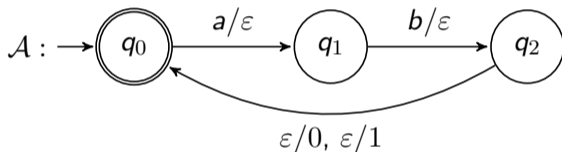
ω -Rational Relations

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- **Example:**



ω -Rational Relations

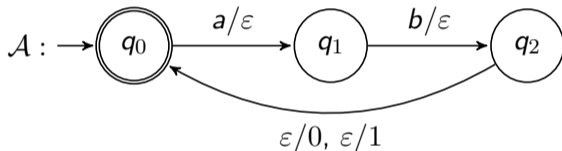
- relations over **infinite** words
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- **Example:**



- **Büchi** acceptance condition (visit an accepting state infinitely often)

ω -Rational Relations

- relations over infinite words
- defined by transducers (non-deterministic multi-tape “automata”)
- Example:



- Büchi acceptance condition (visit an accepting state infinitely often)
- $R_\omega(\mathcal{A}) = \{((ab)^\omega, v) \mid v \in \{0, 1\}^\omega\}$
- e.g. $((ab)^\omega, 0^\omega), ((ab)^\omega, (10)^\omega) \in R_\omega(\mathcal{A})$

The equivalence problem for transducers

$$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$$

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$$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$$

Deciding recognizability for (ω -)automatic relations

$$R \stackrel{?}{=} \bigcup_{i=1}^n L_i \times K_i$$

The Equivalence Problem

	finite words	infinite words
Given:	$R_*(\mathcal{A}) \stackrel{?}{=} R_*(\mathcal{B})$	$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$
transducers \mathcal{A}, \mathcal{B}		

The Equivalence Problem

	finite words	infinite words
Given:	$R_*(\mathcal{A}) \stackrel{?}{=} R_*(\mathcal{B})$	$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$
transducers \mathcal{A}, \mathcal{B}	undecidable	

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Given:	$R_*(\mathcal{A}) \stackrel{?}{=} R_*(\mathcal{B})$	$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$
transducers \mathcal{A}, \mathcal{B}	undecidable	undecidable
deterministic transducers \mathcal{A}, \mathcal{B}	decidable	

The Equivalence Problem

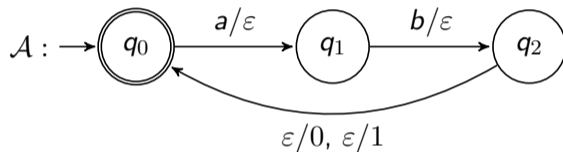
	finite words	infinite words
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transducers \mathcal{A}, \mathcal{B}	undecidable	undecidable
deterministic transducers \mathcal{A}, \mathcal{B}	decidable	undecidable

Theorem (this work)

The equivalence problem for *deterministic Büchi* transducers is *undecidable*.

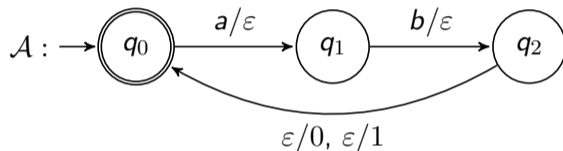
Deterministic (ω -)Rational Relations

deterministic (Büchi) transducers



Deterministic (ω -)Rational Relations

deterministic (Büchi) transducers



- 1 state **determines** the component to be read
- 2 state and letter **determine** the next state

The Equivalence Problem

\mathcal{A}, \mathcal{B} deterministic (Büchi) transducers

Aim

$$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$$

undecidable

The Equivalence Problem

\mathcal{A}, \mathcal{B} deterministic (Büchi) transducers

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$$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$$

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Known

$$R_*(\mathcal{A}) \stackrel{?}{=} R_*(\mathcal{B})$$

decidable

The Equivalence Problem

\mathcal{A}, \mathcal{B} deterministic (Büchi) transducers

Known

$$R_*(\mathcal{A}) \cap R_*(\mathcal{B}) \stackrel{?}{=} \emptyset$$

undecidable

Aim

$$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$$

undecidable

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$$R_*(\mathcal{A}) \stackrel{?}{=} R_*(\mathcal{B})$$

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The Equivalence Problem

\mathcal{A}, \mathcal{B} deterministic (Büchi) transducers

Known	Aim	Known
$R_*(\mathcal{A}) \cap R_*(\mathcal{B}) \stackrel{?}{=} \emptyset$	$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$	$R_*(\mathcal{A}) \stackrel{?}{=} R_*(\mathcal{B})$
undecidable	undecidable	decidable

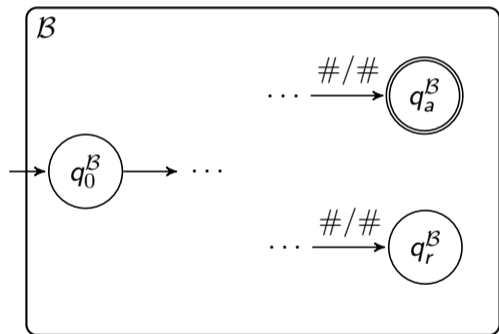
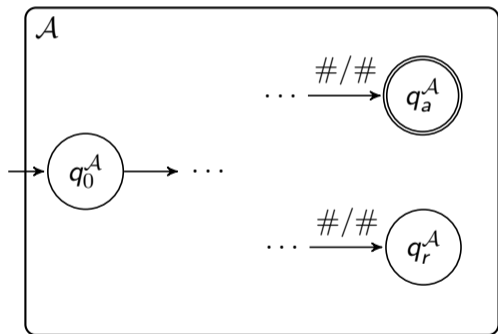
$$(\mathcal{A}, \mathcal{B}) \mapsto (\mathcal{A}', \mathcal{B}')$$

$$\rightsquigarrow R_*(\mathcal{A}) \cap R_*(\mathcal{B}) \neq \emptyset \Leftrightarrow R_\omega(\mathcal{A}') \neq R_\omega(\mathcal{B}')$$

The reduction is based on an idea of Böhm, Göller, Halfon, and Hofman 2017

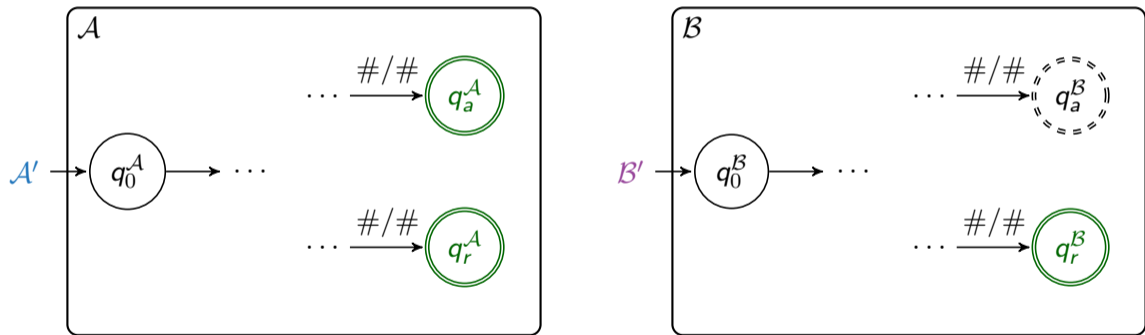
The Equivalence Problem — Reduction

$$R_*(\mathcal{A}) \cap R_*(\mathcal{B}) \neq \emptyset \Leftrightarrow R_\omega(\mathcal{A}') \neq R_\omega(\mathcal{B}')$$



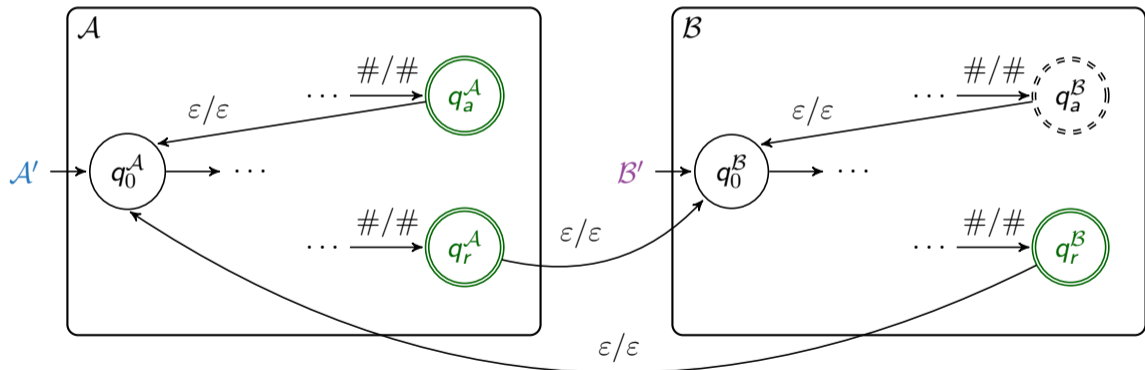
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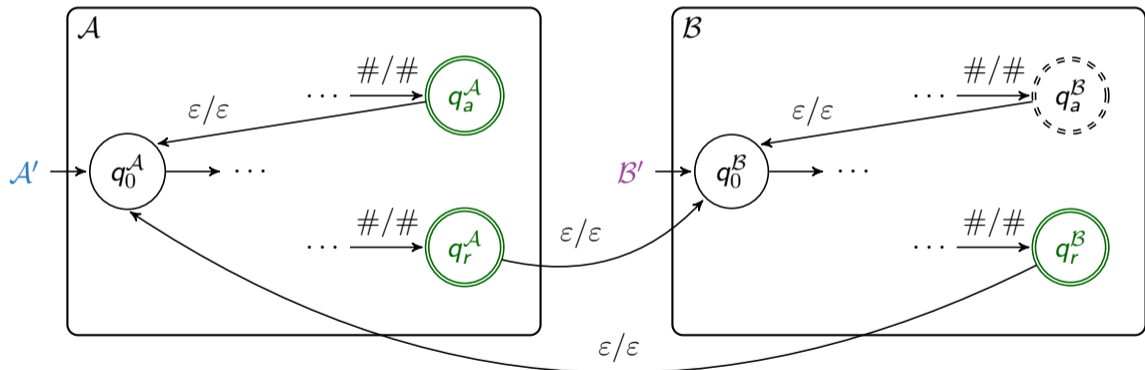
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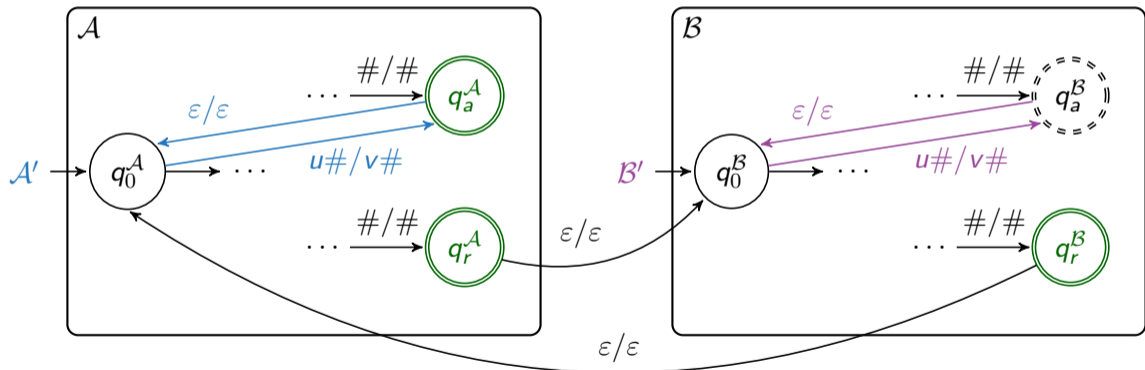
The Equivalence Problem — Reduction

$$(u\#, v\#) \in R_*(\mathcal{A}) \cap R_*(\mathcal{B}) \Rightarrow (u\#, v\#)^\omega \in R_\omega(\mathcal{A}') \setminus R_\omega(\mathcal{B}')$$



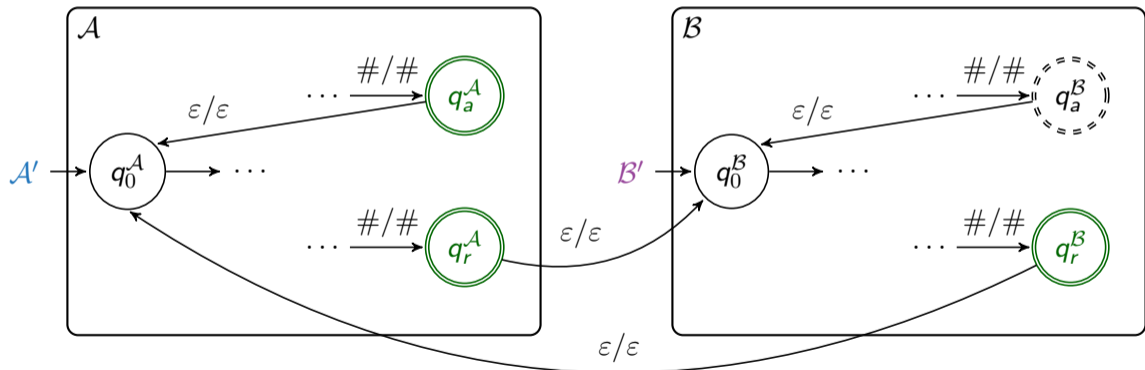
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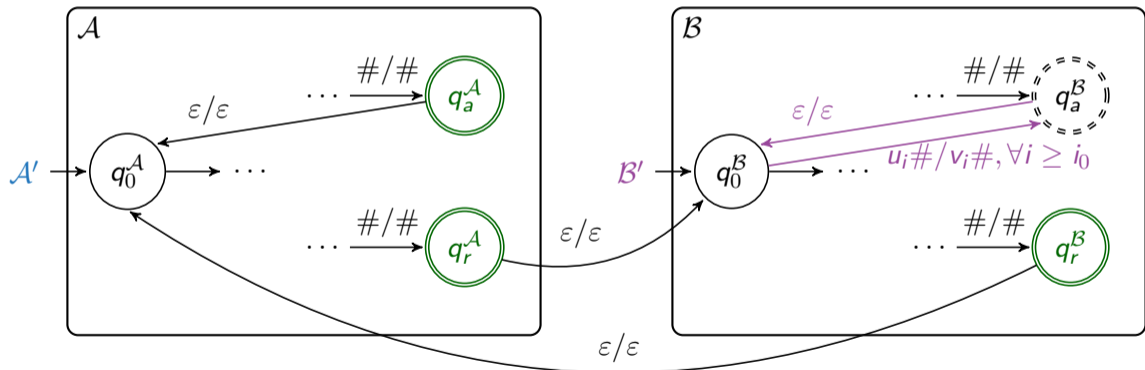
The Equivalence Problem — Reduction

$$(u_1\#, v_1\#)(u_2\#, v_2\#)\dots \in R_\omega(\mathcal{A}') \setminus R_\omega(\mathcal{B}') \Rightarrow \exists k : (u_k\#, v_k\#) \in R_*(\mathcal{A}) \cap R_*(\mathcal{B})$$



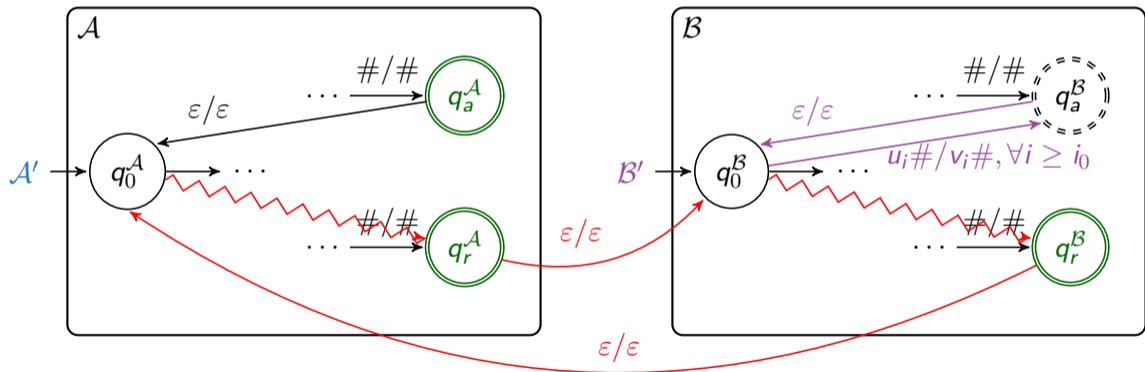
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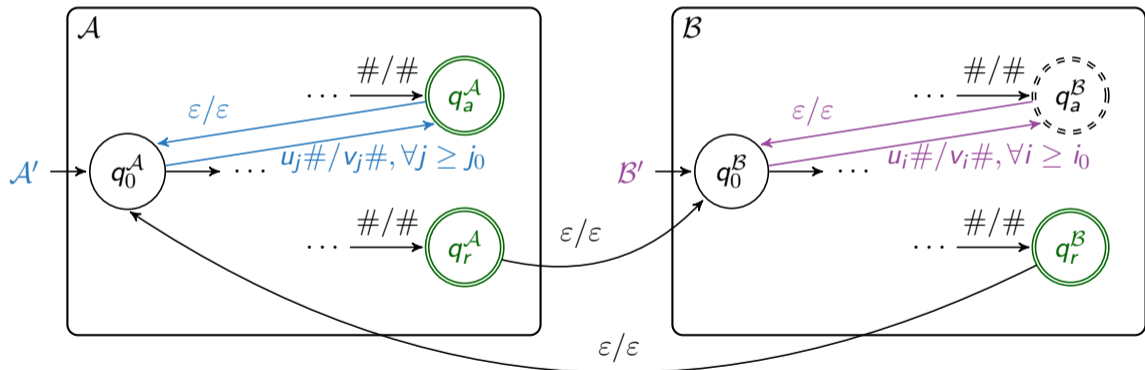
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Outline

The equivalence problem for **deterministic** transducers

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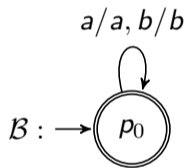
undecidable

Deciding recognizability for $(\omega-)$ automatic relations

$$R \stackrel{?}{=} \bigcup_{i=1}^n L_i \times K_i$$

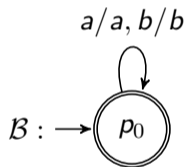
(ω) -Automatic Relations

- **synchronous** transducer (single reading head)
- “automaton over a product alphabet”



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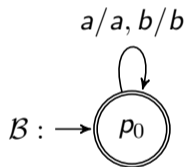


$$R_*(\mathcal{B}) = \{(u, u) \mid u \in \{a, b\}^*\}$$

$$R_\omega(\mathcal{B}) = \{(u, u) \mid u \in \{a, b\}^\omega\}$$

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- **finite** words may be of **different length**

$(\omega-)$ Recognizable Relations

$$R = \bigcup_{i=1}^n L_i \times K_i$$

↑ ↘ ↗ ↑
 $(\omega-)$ regular languages

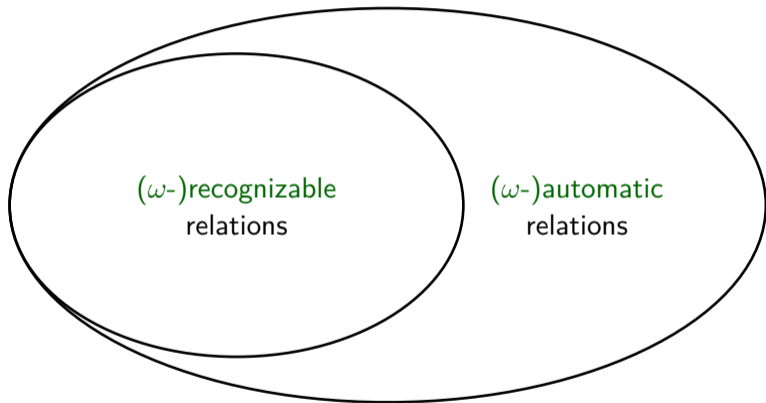
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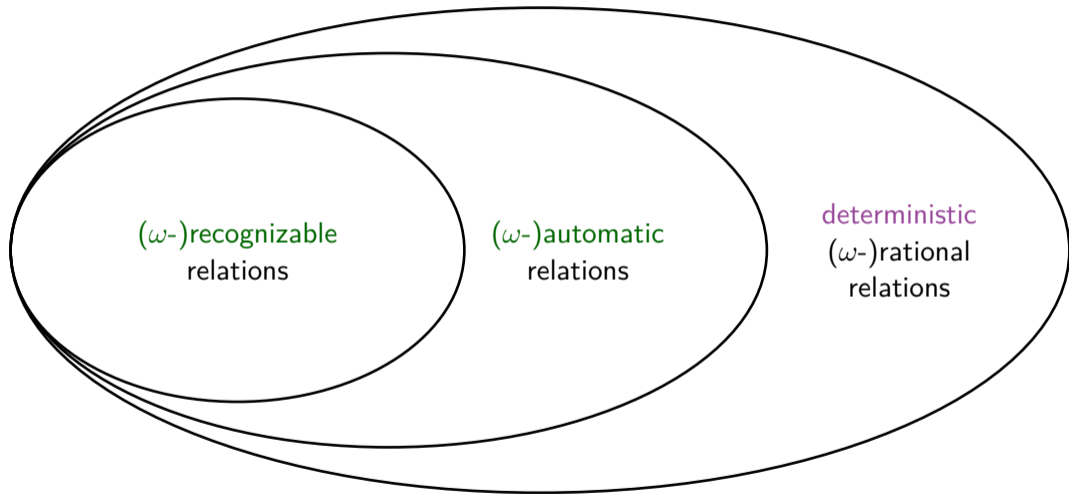
↑ ↘ ↗
 (ω) -regular languages

Example: $R = a^\omega \times a^\omega \cup b^\omega \times (a + b)^\omega$

(ω) -Rational Relations – Hierarchy



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Deciding Recognizability of (ω -)Automatic Relations

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Theorem (Carton, Choffrut, and Grigorieff 2006)

Recognizability for *automatic relations* is *decidable* in *doubly exponential time*.

Deciding Recognizability of (ω -)Automatic Relations

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Theorem (Carton, Choffrut, and Grigorieff 2006)

Recognizability for *automatic relations* is *decidable* in *doubly exponential time*.

Theorem (this work)

Recognizability is *decidable* for

- ① *ω -automatic relations* in *doubly exponential time*,
- ② *binary automatic relations* in *exponential time*.

Summary

The equivalence problem for deterministic transducers

$$R_{\omega}(\mathcal{A}) \stackrel{?}{=} R_{\omega}(\mathcal{B})$$

undecidable

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The equivalence problem for deterministic transducers

$$R_\omega(\mathcal{A}) \stackrel{?}{=} R_\omega(\mathcal{B})$$

undecidable

Deciding recognizability

$$R \stackrel{?}{=} \bigcup_{i=1}^n L_i \times K_i$$

R ω -automatic

2EXPTIME

R binary automatic

EXPTIME

Literature

- Böhm, Stanislav, Stefan Göller, Simon Halfon, and Piotr Hofman (2017). “On Büchi One-Counter Automata”. In: 34th Symposium on Theoretical Aspects of Computer Science (STACS 2017). Ed. by Heribert Vollmer and Brigitte Vallée. Vol. 66. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 14:1–14:13.
- Carton, Olivier, Christian Choffrut, and Serge Grigorieff (2006). “Decision problems among the main subfamilies of rational relations”. In: RAIRO-Theoretical Informatics and Applications 40.02, pp. 255–275.