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Decidable Weighted Expressions with Presburger Combinators

Boolean vs Quantitative Languages

$$L : \Sigma^* \rightarrow \{0, 1\}$$

Classical decision problems

Emptiness	$\exists u. f(u) \geq 1$
Universality	$\forall u. f(u) \geq 1$
Inclusion	$\forall u. f(u) \geq g(u)$
Equivalence	$\forall u. f(u) = g(u)$

Boolean vs Quantitative Languages

$$L : \Sigma^* \rightarrow \{0, 1\} \mathbb{Z} \cup \{-\infty\}$$

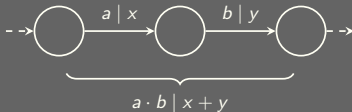
Classical **quantitative** decision problems

Emptiness	$\exists u. f(u) \geq \cancel{\nu}$	for some threshold ν
Universality	$\forall u. f(u) \geq \cancel{\nu}$	for some threshold ν
Inclusion	$\forall u. f(u) \geq g(u)$	
Equivalence	$\forall u. f(u) = g(u)$	

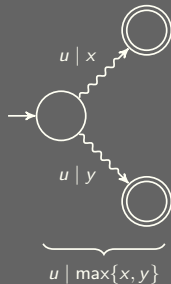
Classical Model: Weighted Automata

$(\max, +)$ WA

Transition sequence



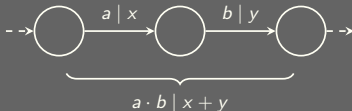
Non-determinism



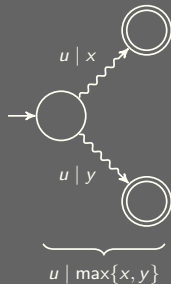
Classical Model: Weighted Automata

(max,+) WA

Transition sequence



Non-determinism



Undecidability [Krob 1994]

- Quantitative language-inclusion is undecidable for (max,+) WA
- ▶ Even for linearly ambiguous automata [Colcombet 2010]

Decidable Formalisms: Restriction

Finitely ambiguous (max,+) WA [Filiot et al. 2012]

Define functions of the form,

$$u \mapsto \max\{\mathcal{A}_1(u), \dots, \mathcal{A}_k(u)\}$$

\mathcal{A}_i : Unambiguous WA

- 😊 Quantitative decision problems are DECIDABLE
- 😊 Closed under *max* and *sum*
- 😞 Limited expressive power (*min*, *minus*, ...)

Decidable Formalisms: New model

Mean-payoff expressions [Chatterjee et al. 2010]

$$E ::= \mathcal{A} \mid \max(E, E) \mid \min(E, E) \mid E + E \mid -E$$

\mathcal{A} : Deterministic WA

- 😊 Quantitative decision problems are PSPACE-COMPLETE [Velner 2012]
- 😊 Closed under *max*, *min*, *sum* and *minus*
- 😞 Determinism (define Lipschitz continuous functions)
- 😞 Does **not** contain all finitely ambiguous (max,+) WA
- 😞 Monolithism (apply on the whole word)

Contributions

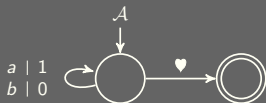
1 Simple expressions

$$E ::= \mathcal{A} \mid \phi(E, E)$$

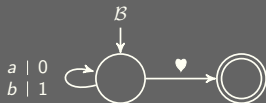
\mathcal{A} : Unambiguous WA

ϕ : $\exists \text{FO}[\leq, +, 0, 1]$ formula defining function with arity two

Example



$$E = \max(\mathcal{A} - \mathcal{B}, \mathcal{B} - \mathcal{A})$$



$$u \mapsto |\mathcal{A}(u) - \mathcal{B}(u)|$$

Contributions

1 Simple expressions

$$E ::= \mathcal{A} \mid \phi(E, E)$$

\mathcal{A} : Unambiguous WA

ϕ : $\exists \text{FO}[\leq, +, 0, 1]$ formula defining function with arity two

- 😊 Quantitative decision problems are PSPACE-COMPLETE
- 😊 Closed under Presburger definable functions
- 😊 Contain all finitely ambiguous $(\max, +)$ WA
- 😞 Monolithism (apply on the whole word)

Contributions

2 Iterable expressions

$$E ::= \mathcal{A} \mid \phi(E, E) \mid E^{\circledast}$$

- ▶ Sum arbitrarily many factors
- ▶ Unique decomposition required

The diagram illustrates the decomposition of an iterable expression $E^{\circledast}(u)$. At the top, a horizontal line represents the domain u , which is partitioned into segments u_1, \dots, u_n by vertical tick marks. Below this, a large box represents the expression $E^{\circledast}(u)$. Inside this box, the segments u_1, \dots, u_n are mapped to the expression $E(u_1) + \dots + E(u_n)$. The plus signs and ellipsis are in red, and the entire expression is enclosed in a rectangular frame.

Examples

$$E^{\circledast}$$

$$u_1 \heartsuit u_2 \heartsuit \dots \heartsuit u_n \heartsuit \mapsto \sum_{i=1}^n E(u_i)$$

$$\phi(E^{\circledast}, F^{\circledast})$$

$$u \mapsto \phi \left\{ \sum_{i=1}^n E(u_i), \sum_{j=1}^m F(v_j) \right\}$$

Results

Theorem (Iterable Expressions)

Quantitative decision problems are UNDECIDABLE



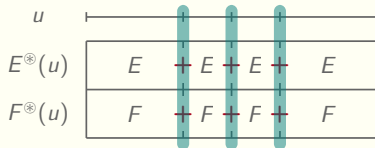
Results

Theorem (Iterable Expressions)

Quantitative decision problems are UNDECIDABLE

Theorem (Synchronised Iterable Expressions)

Quantitative decision problems are DECIDABLE



Results

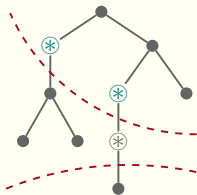
Theorem (Iterable Expressions)

Quantitative decision problems are UNDECIDABLE

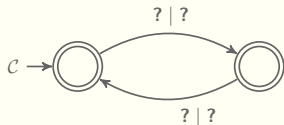
Theorem (Synchronised Iterable Expressions)

Quantitative decision problems are DECIDABLE

Synchronisation property is PTIME



Weighted Chop Automata

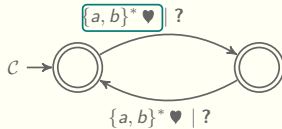


New model

- ▶ Generalise unambiguous WA
- ▶ Recursive definition

Weighted Chop Automata

Regular language

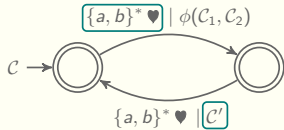


New model

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Weighted Chop Automata

Regular language



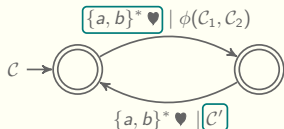
Presburger formula
use sub-WCA

New model

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Weighted Chop Automata

Regular language



Presburger formula
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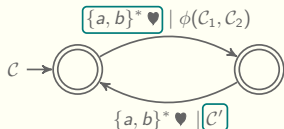
- ▶ Generalise unambiguous WA
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Example

$$C(aab \heartsuit baa \heartsuit) = \phi(C_1(aab \heartsuit), C_2(aab \heartsuit)) + C'(baa \heartsuit)$$

Weighted Chop Automata

Regular language



Presburger formula
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- ▶ Generalise unambiguous WA
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Example

$$C(aab \heartsuit baa \heartsuit) = \phi(C_1(aab \heartsuit), C_2(aab \heartsuit)) + C'(baa \heartsuit)$$

Operators for expressiveness equivalence

$$\left. \begin{array}{l} E \odot F : u_1 u_2 \mapsto E(u_1) + F(u_2) \\ E \triangleright F : u \mapsto \text{if } u \in \text{dom}(E) \text{ then } E(u) \text{ else } F(u) \end{array} \right\} [\text{Alur 2014}]$$

Conclusion

Summary

Simple expressions: PSPACE-COMPLETE

Sum-iterable expressions: UNDECIDABLE

Synchronised sum-iterable expressions: DECIDABLE

Perspective

Iterate other operations (*max*, Presburger definable functions, ...)

Thanks!

Conclusion

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Perspective

Iterate other operations (*max*, Presburger definable functions, ...)

$$\begin{array}{ccccccc} rrrrg & rrrrrrrrg & \dots & rrrrrrg \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ 4 & 7 & r^n g \mapsto n & 5 \end{array}$$

└──┘
iterate max

Thanks!