

Token Jumping on minor-closed classes

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FCT'17



Reconfiguration of Independent Sets

Introduced by Hearn and Demaine in 2005
in a general study of one-player games :

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- Applications to **sampling**.
- Reconfiguration introduced for colorings, satisfiability problems, dominating sets, cliques, list colorings, bases of matroids...

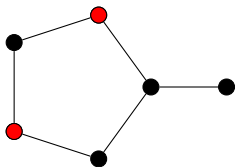
Main questions

- **Reachability problem.** Given two configurations, is it possible to **transform** the one into the other?
- **Connectivity problem.** Given **any pair** of configurations, is it possible to transform the one into the other?
- **Minimization.** Given two configurations, what is the length of a **shortest** sequence?

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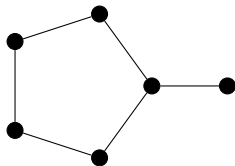
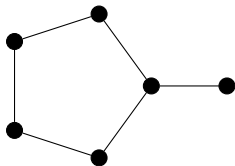
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- **Minimization.** Given two configurations, what is the length of a **shortest** sequence?
- **Algorithmics.** Can we **efficiently decide** these problems? (In polynomial time, FPT-time...).

Independent Sets



A (k) -independent set is a subset of vertices (of size k) that are pairwise non incident.

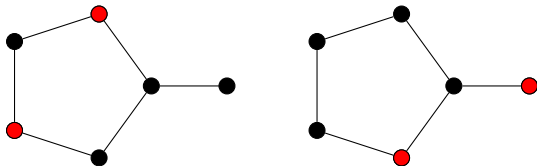
Token Jumping



Definition (TJ-sequence)

A TJ-sequence I_1, \dots, I_ℓ of independent sets is a sequence such that there exist $v \in I_{j+1}$ and $u \in I_j$ such that $I_{j+1} = I_j \cup \{v\} \setminus \{u\}$.

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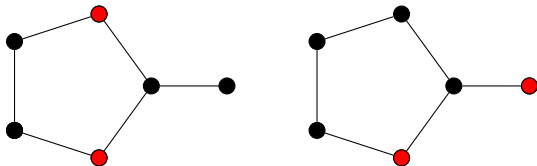
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Equivalent formulation :

We put a token on each vertex of the Independent Set and we can **move tokens** one by one in such a way the set remains an independent set at any step.

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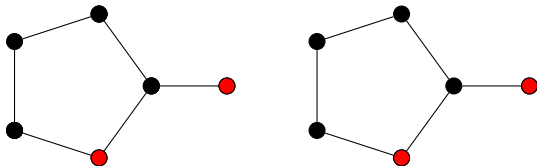
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Input : A graph G , $k \in \mathbb{N}$, two independent sets I, J of size k .

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PSPACE-complete on :

- [Hearn, Demaine] Planar graphs of degree ≤ 3 .
- [Kamiński et al.] Perfect graphs.
- [Wrochna] Bounded bandwidth graphs.

Parameterized complexity

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A problem parameterized by k is **FPT** (Fixed Parameter Tractable) if it can be decided in time $Poly(n) \cdot f(k)$.

W[1]-hard problem : Under algorithmic assumption, no FPT algorithm.

Our result

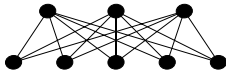
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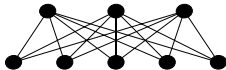
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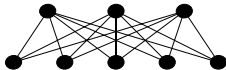


Question (Ito et al.)

Is TJ-reachability FPT on $K_{\ell,\ell}$ -free graphs parameterized by k ?

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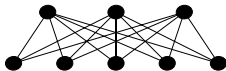


Theorem (B., Mary, Parreau)

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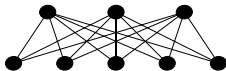
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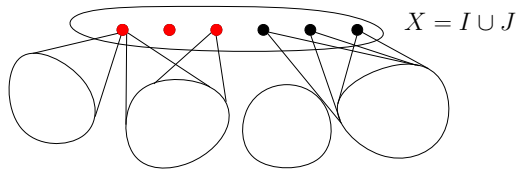
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Corollary :

FPT on bounded treewidth, bounded degeneracy, minor closed (etc...) classes of graphs.

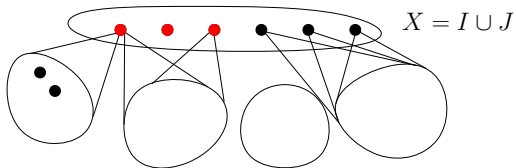
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- Partition G into classes with same neighborhood in $X = I \cup J$.
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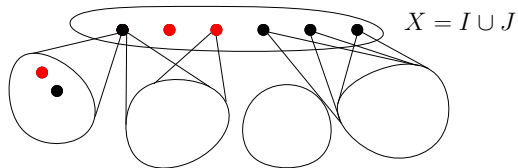
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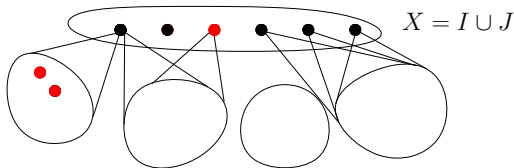
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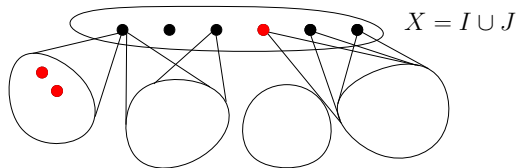
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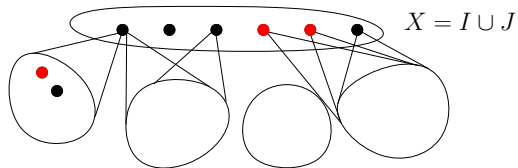
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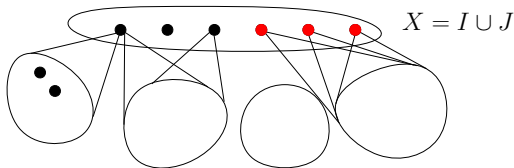
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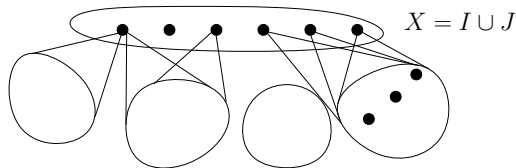
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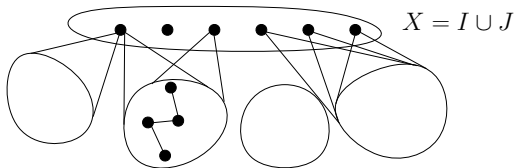
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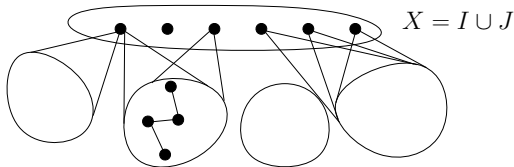
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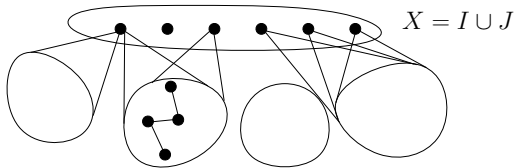
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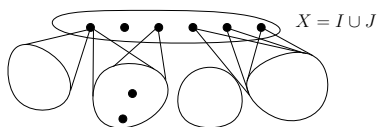
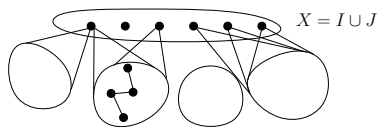
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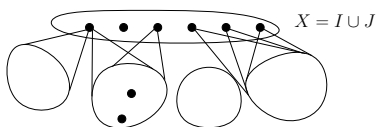
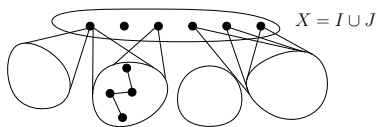
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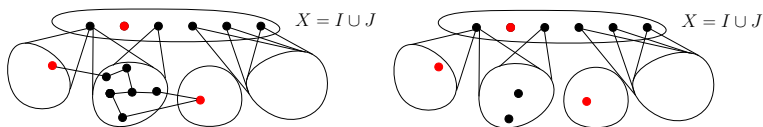
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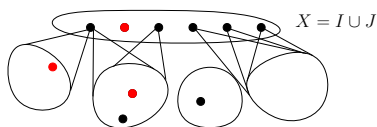
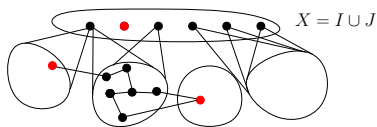
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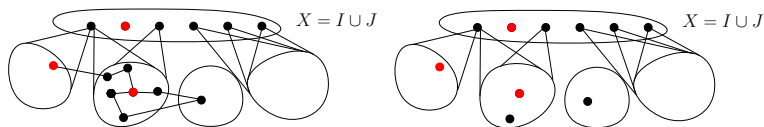
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Idea : Modify X to include these special vertices.

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Technical lemma :

After $\theta(k)$ iterations \Rightarrow Replace what remains of the class into an IS of size k .

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