Listing all fixed-length simple cycles in sparse graphs in optimal time

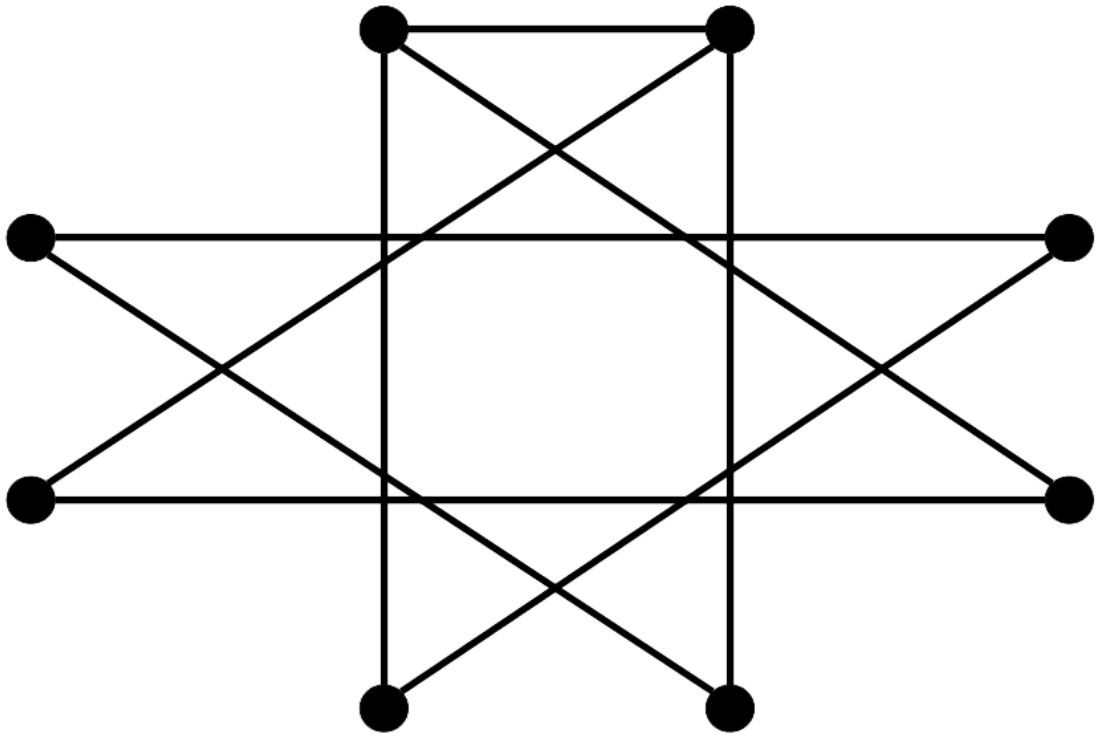
George Manoussakis

Université Paris Saclay, Université Paris Sud, LRI-CNRS, France

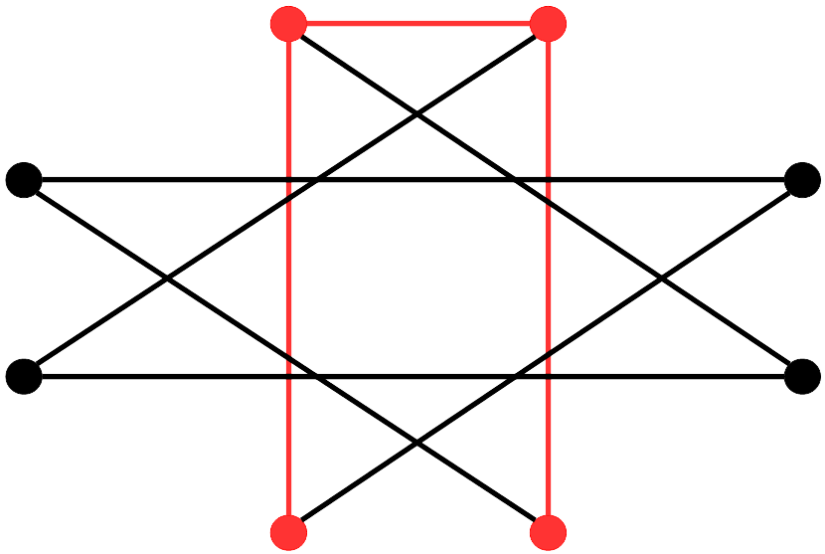
Introduction

Input object: *a graph*

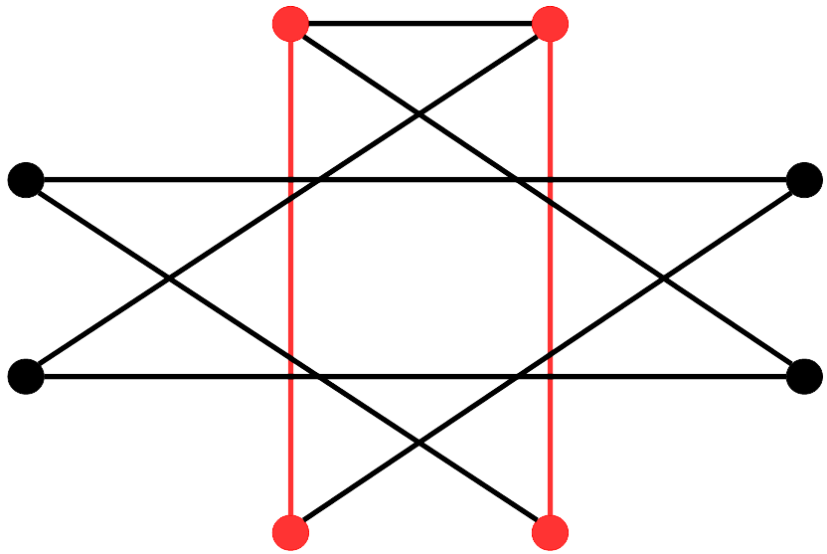
$G = (V, E)$ a graph: $|V| = n$ vertices and $|E| = m$ edges.



Question: find all (induced) subgraphs with some property

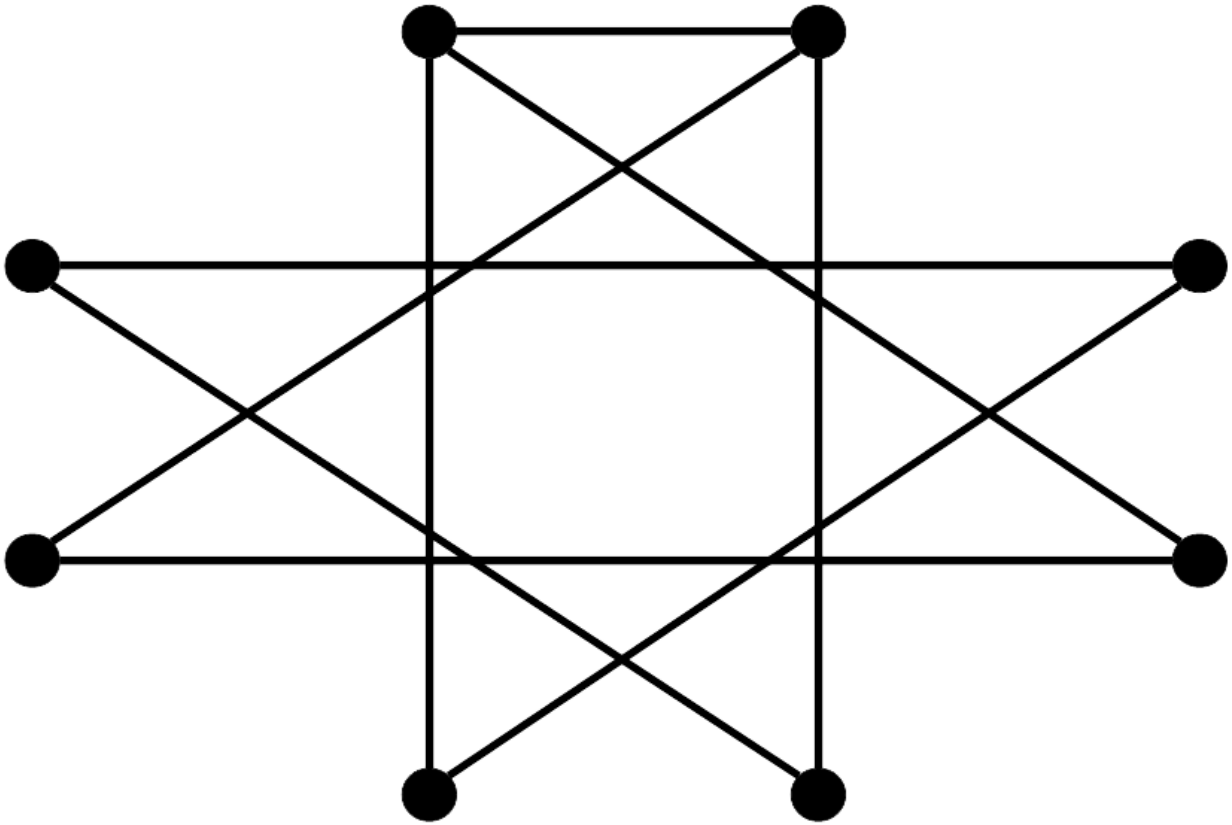


induced subgraph



subgraph

Example: find all cycles of length four



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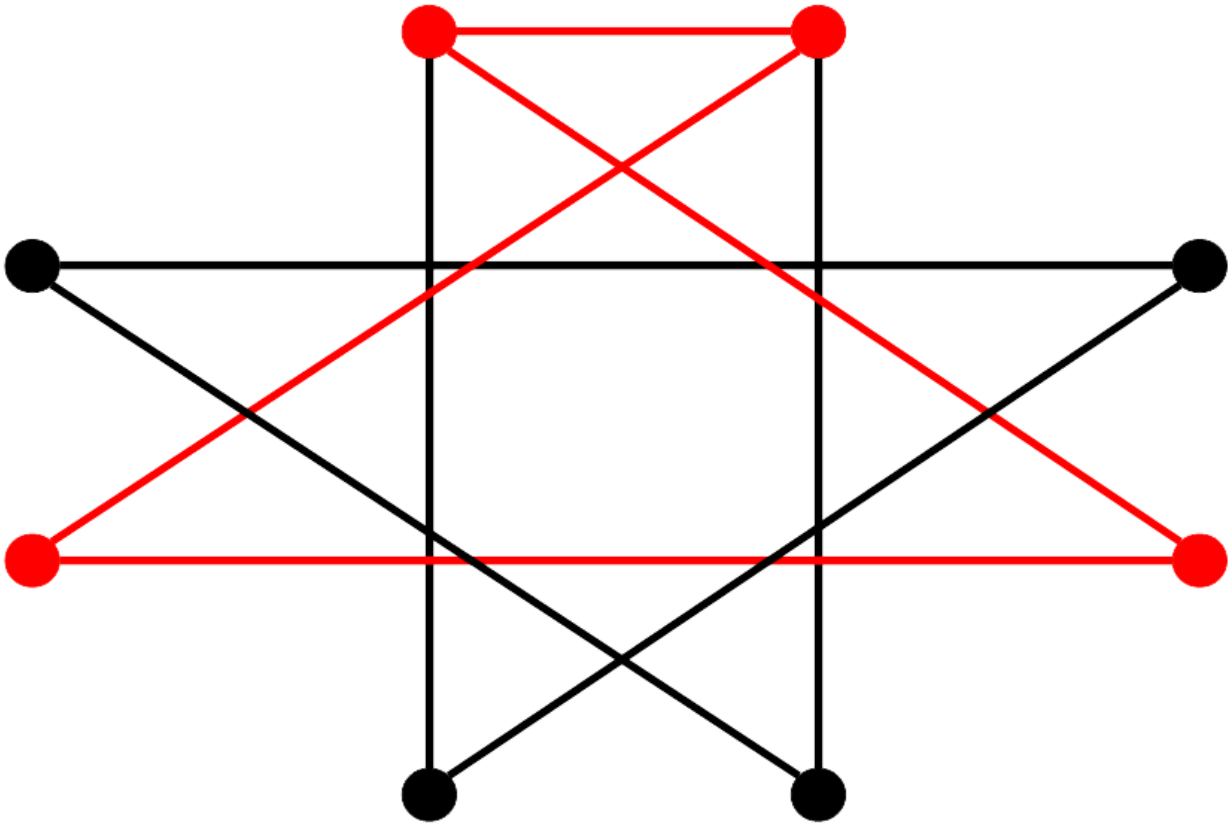

Introduction - k-degenerate graphs

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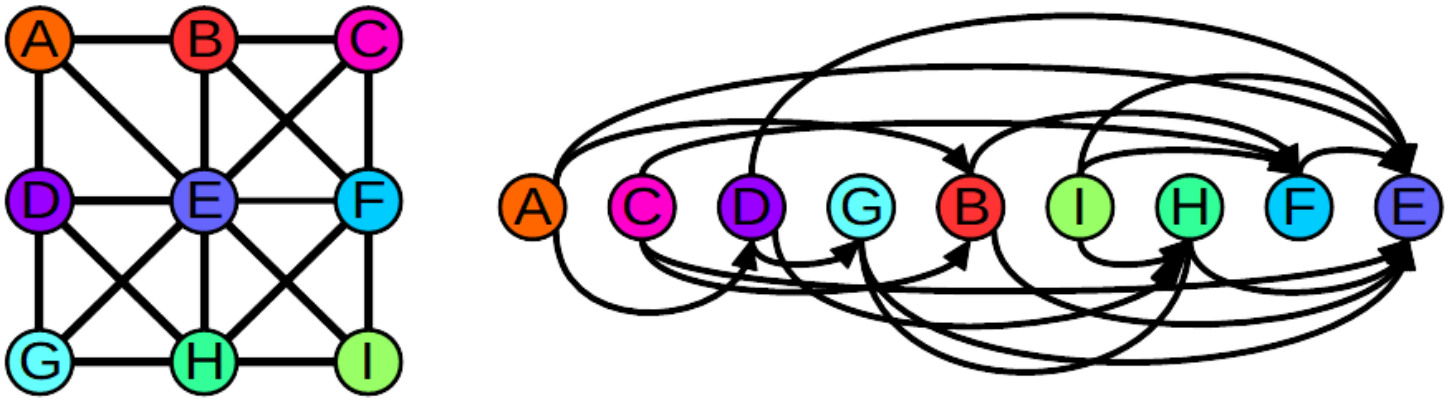

Figure: A 3-degenerate graph and a 3-bounded orientation

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theoretical/practical

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$\forall k, p$ and $n \geq kp, \exists$ n-order k-degenerate graph with $\Omega(n^{\lfloor p/2 \rfloor} k^{\lceil p/2 \rceil})$ p-cycles.

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space optimal:

only requires memory needed to store the graph

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- C_5 and C_6 in planar in $\mathcal{O}(n \log n)$ [Richardson *et al.* '86]

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- C_5 and C_6 in planar in $\mathcal{O}(n \log n)$
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- Any length general and degenerate graphs
  [Alon *et al.* '97]
  for induced cycles : [Cai *et al.* '06]

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this talk: deterministic worst case output size optimal
for any length in degenerate graphs in $\mathcal{O}(n^{\lfloor p/2 \rfloor} k^{\lceil p/2 \rceil})$

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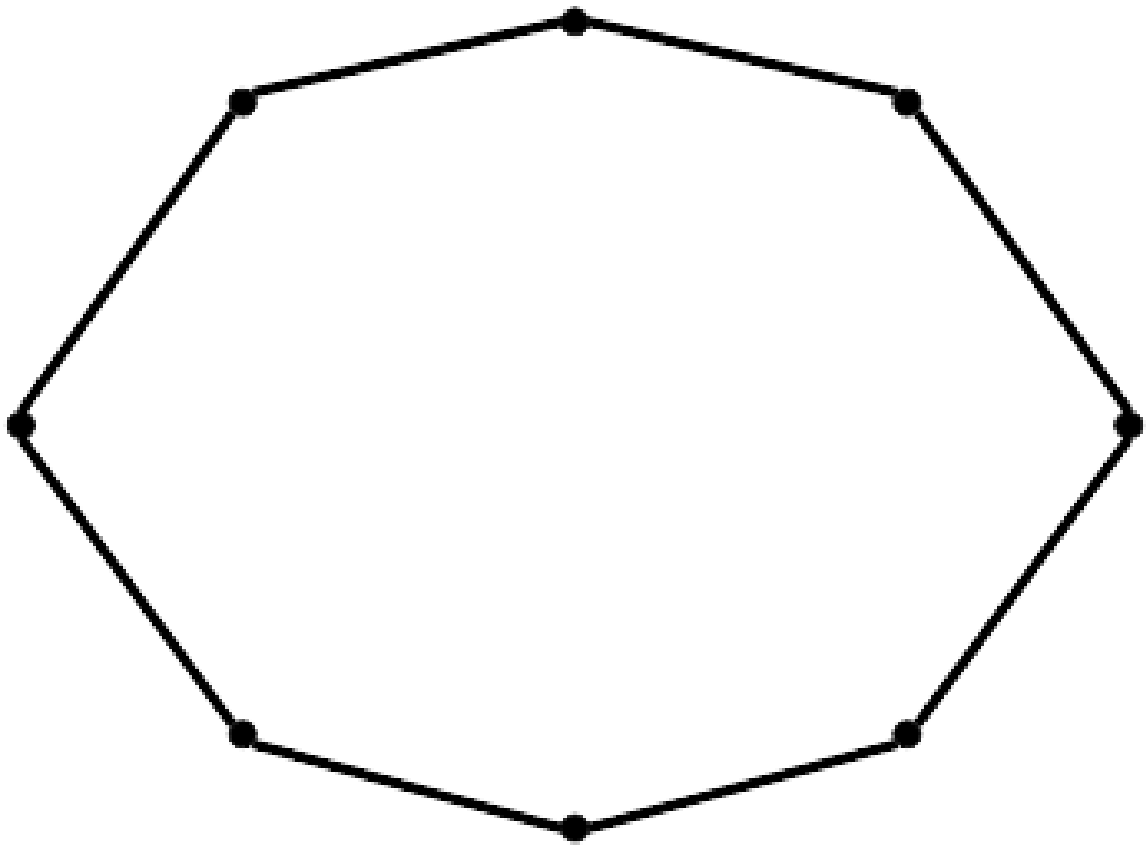
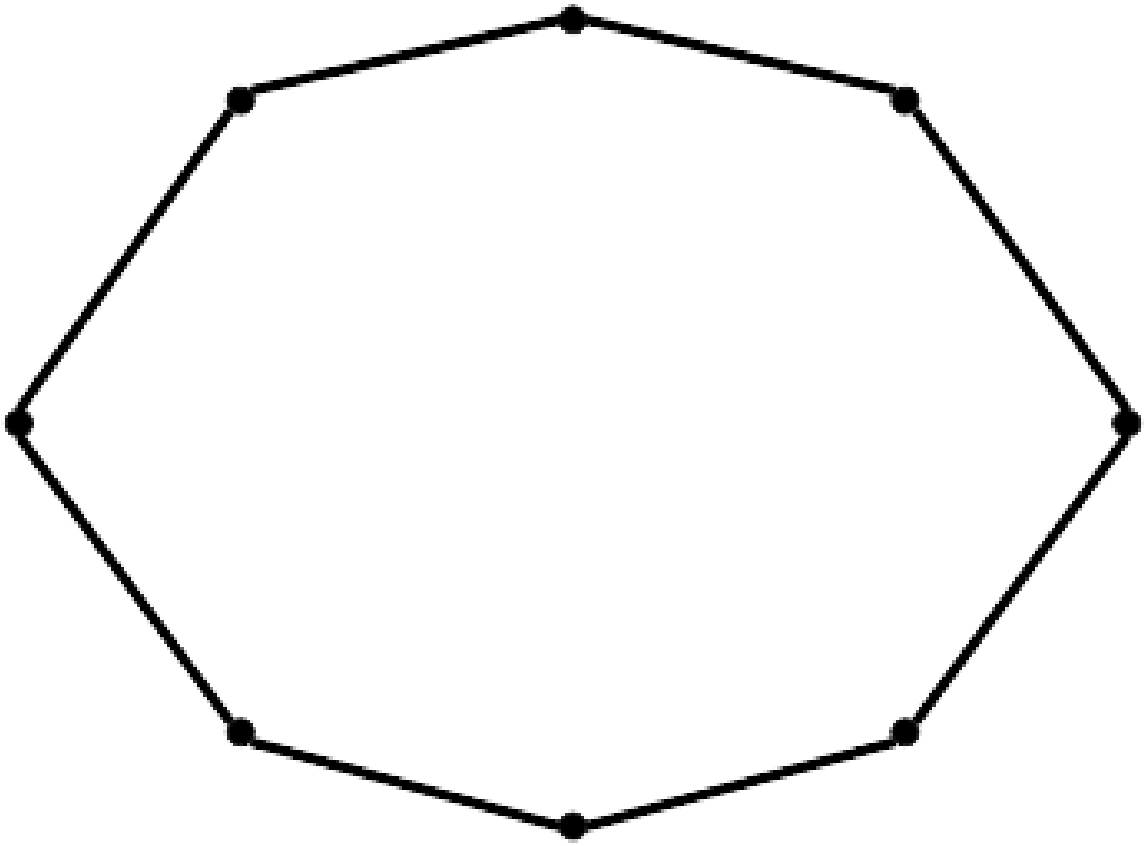
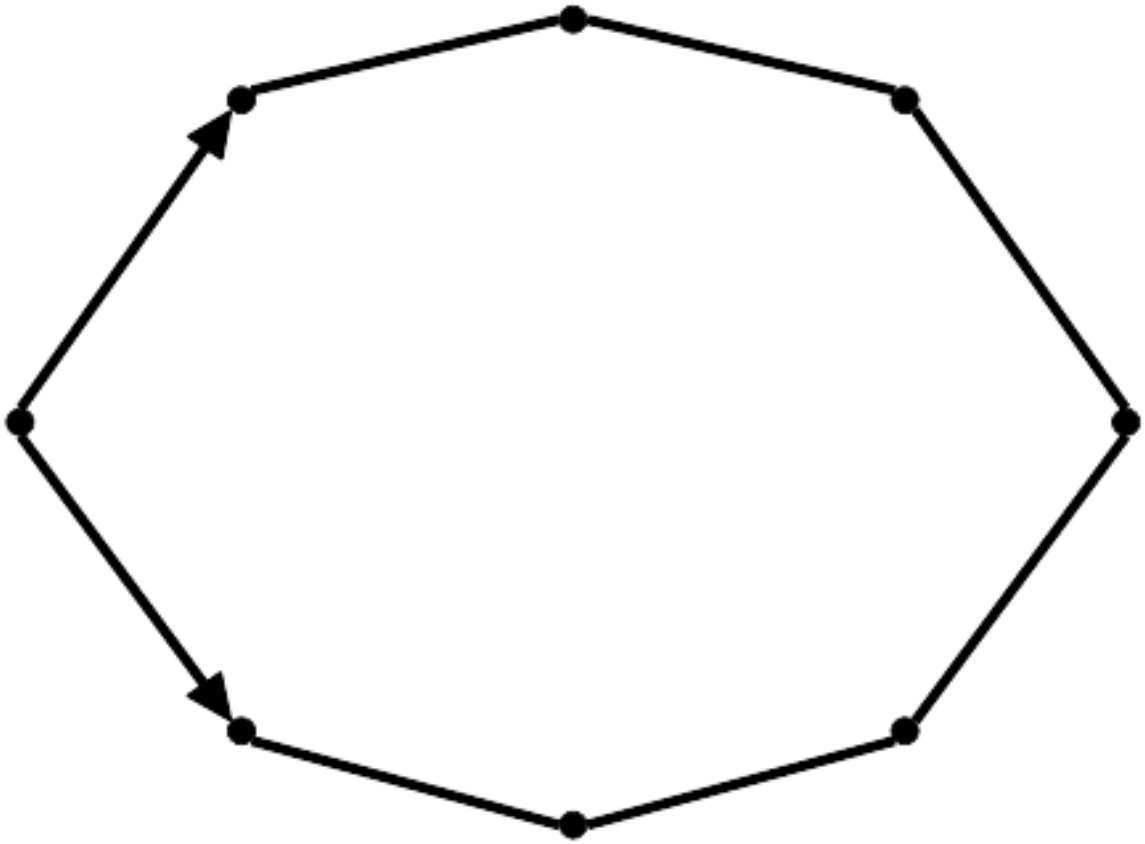
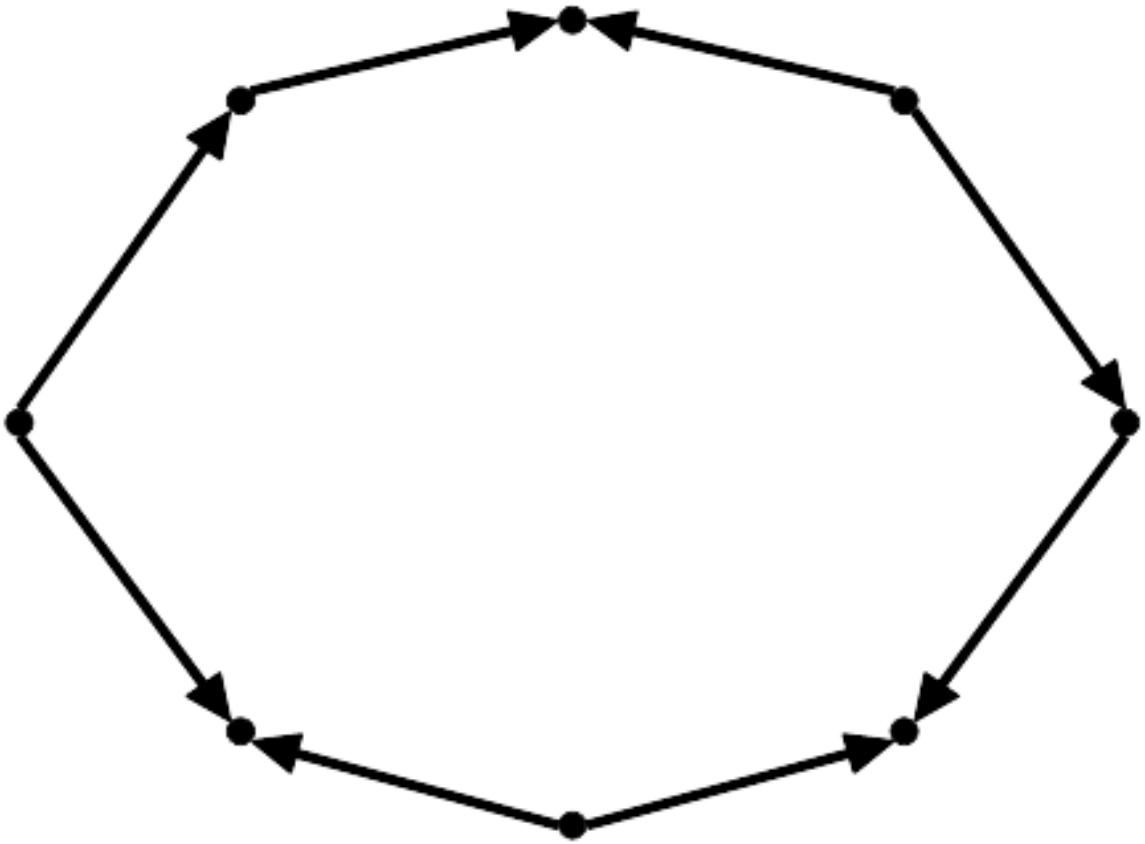
- Form all possible cycles by computing all these possible parts and their possible combinations
- Using properties of the decomposition output each cycle exactly once

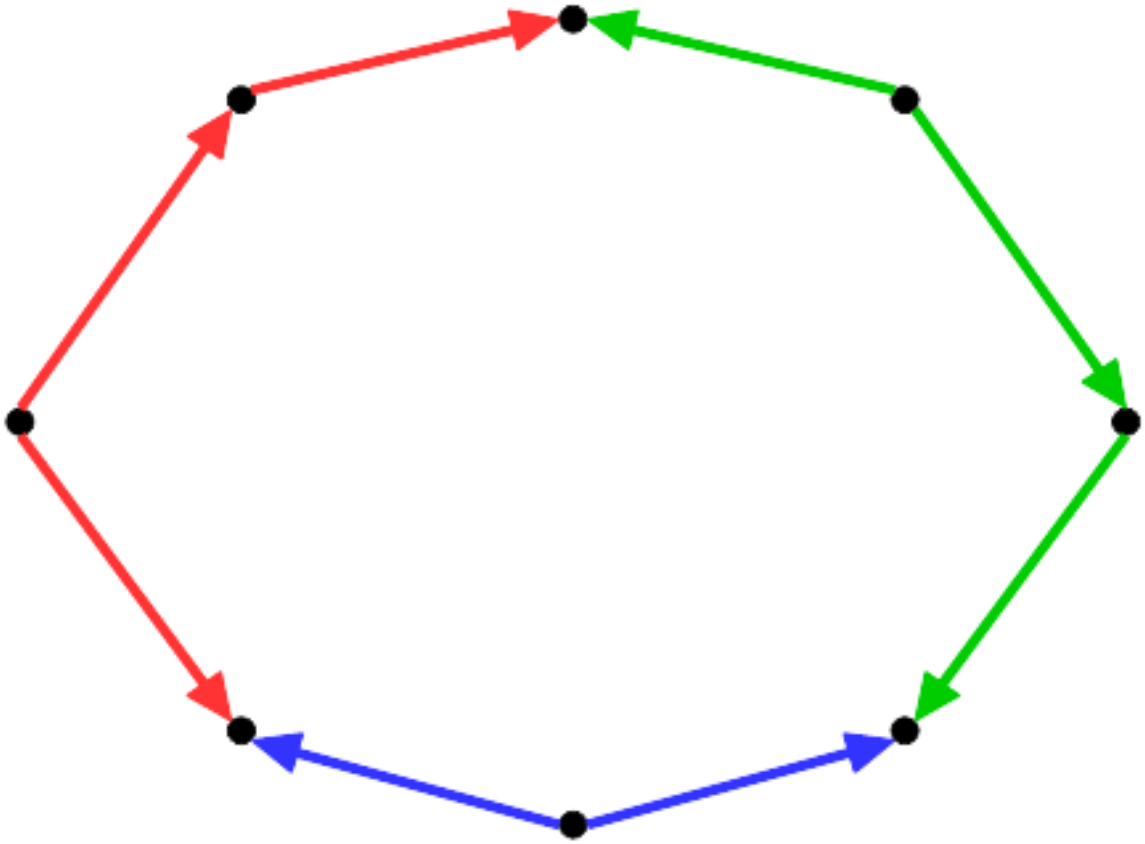




Acyclic orientation: $\exists$ some vertex with two out-going edges







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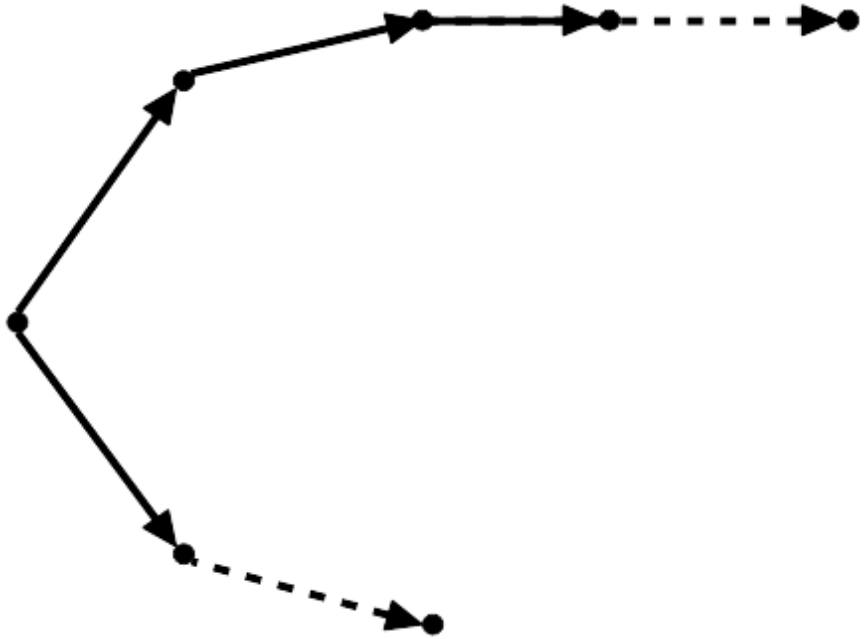
t-path of size (i, j) : two oriented paths of length i and j starting from the same vertex



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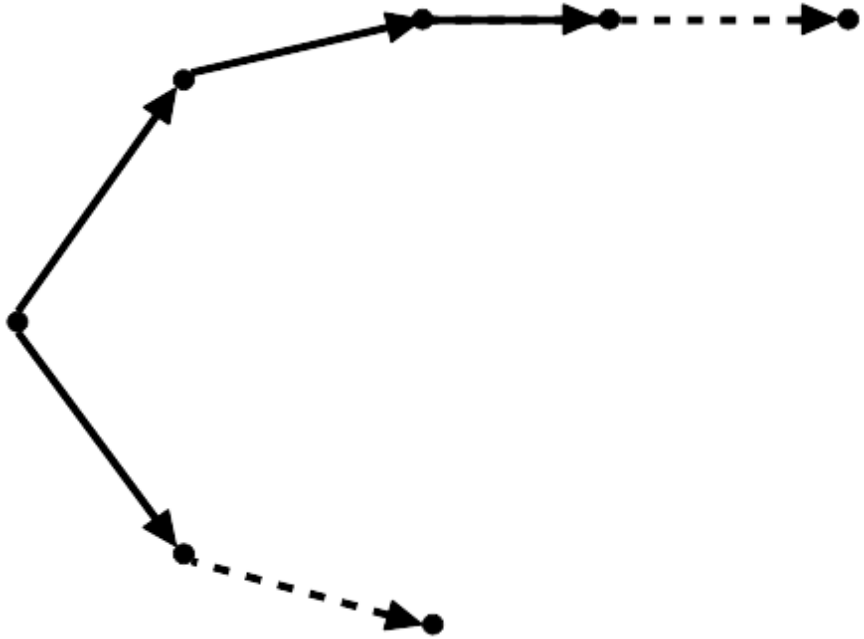

Remark

in k-degenerate graphs, at most nk^{i+j} such paths

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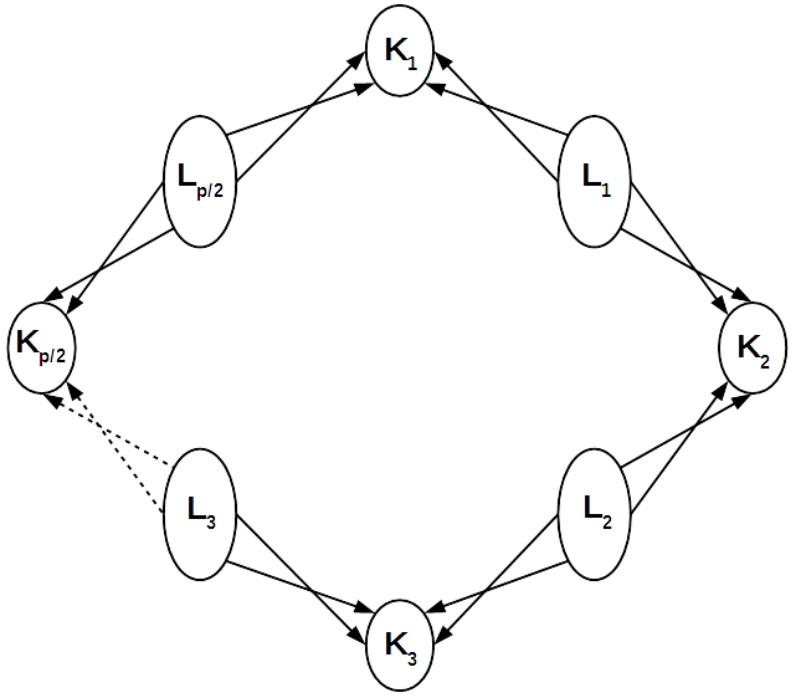
$+$ fine tuning $\implies \mathcal{O}(n^{\lfloor p/2 \rfloor} k^{\lceil p/2 \rceil})$

Proof of optimality: (for k, p even)



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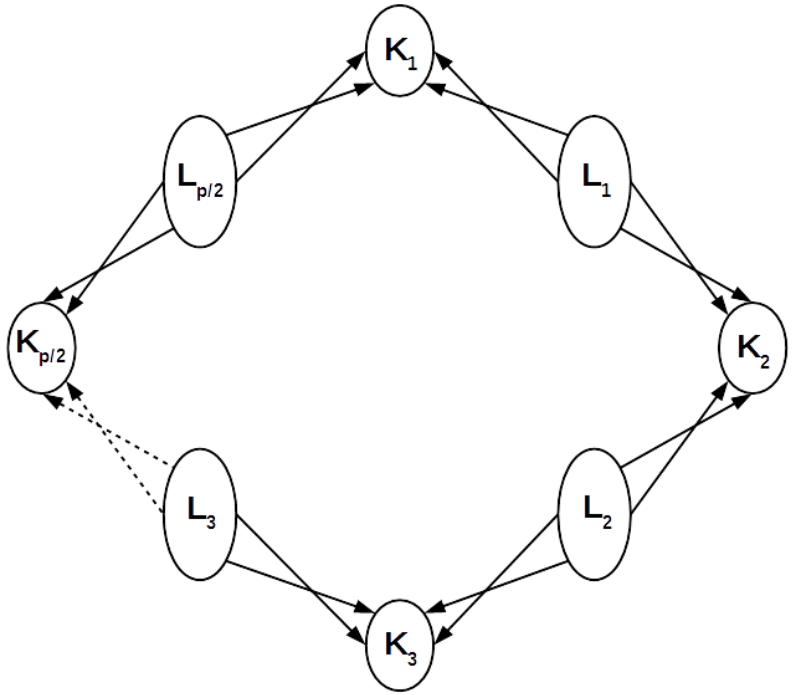

Figure: sets K_i of size k/2, sets L_i of size $h \geq k$

graph is k-degenerate

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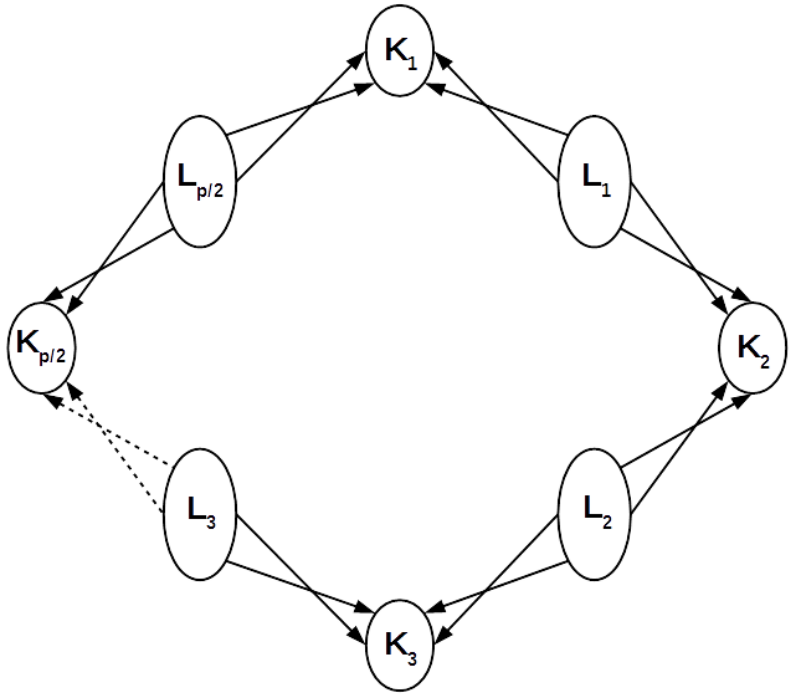

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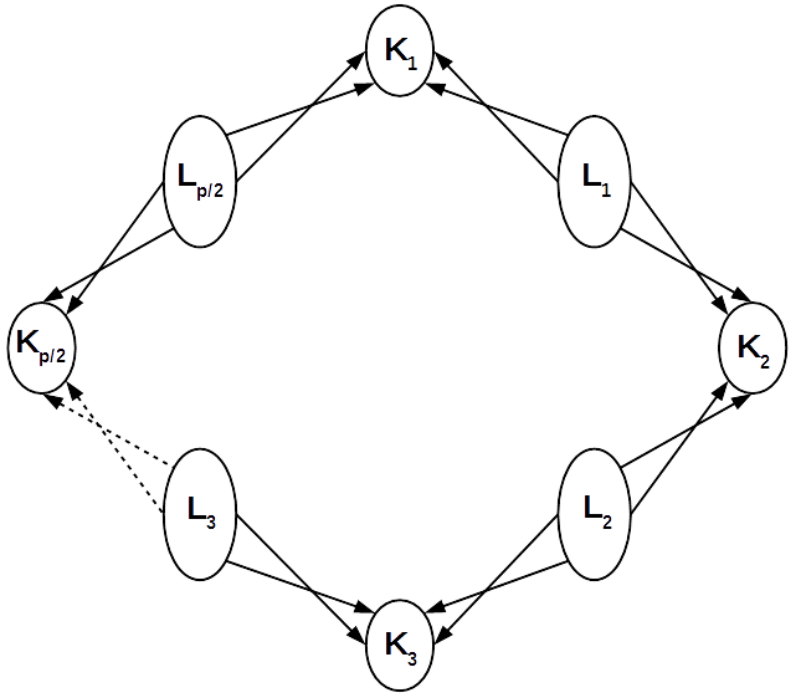

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number of cycles: $(\frac{k}{2})^{p/2} h^{p/2} = \Omega(k^{p/2} n^{p/2})$

Open problems:

- List p-cycles in k-degenerate graphs in $\mathcal{O}(occurences)$?
- *enumerating* p-cycles takes time $\mathcal{O}(n^{\lfloor p/2 \rfloor} k^{\lceil p/2 \rceil})$
  how much faster is *counting*
  - for general graphs:
    $\mathcal{O}(n^{\omega})$ [Alon *et al.* '97] for cycles length ≤ 7
    $\mathcal{O}(n^{\lceil p/2 \rceil + 3})$ [Vassilevska *et al.* '09]
    $\mathcal{O}(n^{0.45p + \mathcal{O}(1)})$ [Bjorklund *et al.* '14]
- what if we do not print cycles ?
  $\mathcal{O}(nk^2)$ for C_4, [Chiba *et al.* '85]
  $\mathcal{O}(nk^3)$ for C_5, [Kowalik '03]
- optimal complexity is attained assuming adjacency matrix, if only adjacency list then $\mathcal{O}(n^{\lfloor p/2 \rfloor} k^{\lceil p/2 \rceil} \log k)$