

Weak space over complex numbers

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Computation over Real and Complex Numbers

Motivation: To provide theoretical foundation for characterizing intrinsic nature of numerical computations over real/complex numbers.

The Blum-Shub-Smale Model

A BSS machine M over \mathbb{F} ($\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$) can be viewed as a Turing machine where a complex (real) number stored in each cell.

- M has a finite number of constants $\alpha_1, \dots, \alpha_k \in \mathbb{F}$, called the parameters of M .
- M can perform $+$, \times , $-$ and \div operations with full precision where the operands are either the contents of the cells or the parameters.
- M can compare content of any cell with 0 ($= 0$ test) and branch based on the result of the comparison.
- Input is an element from $\mathbb{F}^* = \bigcup_{n \geq 0} \mathbb{F}^n$.
- Output is 0 or 1.

The Blum-Shub-Smale Model

Definition

$P_{\mathbb{F}}$ is the set of all subsets of \mathbb{F}^* that can be computed by polynomial time bounded BSS machines.

$NP_{\mathbb{F}}$ is the set of all subsets of \mathbb{F}^* that can be computed by polynomial time bounded *non-deterministic* BSS machines

Time vs Space

- Theory of time complexity is well established (notion of completeness, relativized computation, parallel complexity etc.,.)
- Separations of time complexity classes is known ($NC_{\mathbb{R}} \neq P_{\mathbb{R}}$).
- Defining a feasible notion for space is challenging and widely open.

Notions of space for the BSS model

Known Measures of space:

- Unit cost space: Content of the cell (a value from \mathbb{R} or \mathbb{C}) counted as one unit of space.
- Weak space: a more careful cost model for the contents each cell.

How to count space?

Unit cost model : Count as one cell per non-blank cell of the BSS machine.

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Features :

- Counts the maximum number of cells used during any point of the computation.
- Represents *width* of the algebraic circuit computing the same language.

Limitations :

- Does not measure the size of individual cells.
- Number of configurations is infinite, hence no feasible comparison with time complexity.

Unit cost : too strong

- Each cell in the tape can hold a value from \mathbb{R} .
- A step can perform any arithmetic operations or compare two real numbers.

Theorem (Michaux '89)

Any set $L \subseteq \mathbb{R}^$ that is computable using the BSS model can also be computed by a BSS machine that uses only a constant number of cells.*

Constant space is enough for any computation!!

Weak space

Introduced by Naurois '06.

M be a BSS machine with parameters $\alpha_1, \dots, \alpha_k$.

- On a given input $x \in \mathbb{F}^n$, at any stage of computation, every cell \mathbf{c} of M represents a rational function
$$f_{\mathbf{c}} = p_{\mathbf{c}}(x_1, \dots, x_n, \alpha_1, \dots, \alpha_k) / q_{\mathbf{c}}(x_1, \dots, x_n, \alpha_1, \dots, \alpha_k).$$
- For the term $c_{\gamma} x_1^{\gamma_1} \dots x_n^{\gamma_n} \alpha_1^{\gamma_{n+1}} \dots \alpha_k^{\gamma_{n+k}}$, space
$$\approx \log c_{\gamma} + \sum_{i=1}^{n+k} \log \gamma_i.$$
- For the polynomial $p = \sum_{\gamma} c_{\gamma} X^{\gamma}$

$$\text{space}(p) = \sum_{\gamma, c_{\gamma} \neq 0} \log c_{\gamma} + \sum_{i=1}^{n+k} \log \gamma_i.$$

Weak space

- For a rational function

$$f_{\mathbf{c}} = p_{\mathbf{c}}(x_1, \dots, x_n, \alpha_1, \dots, \alpha_k) / q_{\mathbf{c}}(x_1, \dots, x_n, \alpha_1, \dots, \alpha_k),$$

$$space(f_{\mathbf{c}}) = space(p_{\mathbf{c}}) + (q_{\mathbf{c}}).$$

- for a configuration Γ with non-empty cells $\mathbf{c}_1, \dots, \mathbf{c}_m$,

$$space(\Gamma) = \sum_{j=1}^m space(\mathbf{c}_j).$$

- Space of M on x is the max of all configurations.
- Gives a reasonable definition of $LOGSPACE_W$ and $PSPACE_W$.

Theorem (Naurois 06)

- $\text{LOGSPACE}_W \subseteq P_W \cap \text{NC}_{\mathbb{R}}^2$.
- $\text{PSPACE}_W \subseteq P_{\mathbb{R}}$

Conjecture (Naurais 06)

1. $\text{NC}_{\mathbb{R}}^1 \not\subseteq \text{LOGSPACE}_W$.
2. $\text{LOGSPACE}_W \subseteq \text{NC}_{\mathbb{R}}^1 \implies \text{DLOG} \subseteq \text{NC}^1$.

Theorem (1)

$NC_{\mathbb{C}}^1 \not\subseteq PSPACE_W$, i.e., there is a set $L \in NC_{\mathbb{C}}^1$ but $L \notin PSPACE_W$.

Our Results - 2

For a complexity class $\mathcal{C} \subseteq \mathbb{F}^*$, let $\text{BP}(\mathcal{C}) = \mathcal{C} \cap \{0, 1\}^*$.

We prove:

Theorem (2)

$\text{BP}(\text{LOGSPACE}_W) \subseteq \text{DLOG}$.

Proof sketch

Theorem (1)

$\text{NC}_{\mathbb{C}}^1 \not\subseteq \text{PSPACE}_W$, i.e., there is a set $L \in \text{NC}_{\mathbb{C}}^1$ but $L \notin \text{PSPACE}_W$.

Sketch.

- Let $\text{Sym}_{n,n/2}(x_1, \dots, x_n) = \sum_{S \subseteq [n], |S|=n/2} \prod_{i \in S} x_i$.
- Let $L_n = \{(x_1, \dots, x_n) \mid \text{Sym}_{n,n/2}(x_1, \dots, x_n) = 0\}$, and $L = \bigcup_{n \geq 0} L_n$.
- $L \in \text{NC}_{\mathbb{C}}^1$. We show that $L \notin \text{PSPACE}_W$.



Lemma

Let $L \in \text{Space}(s(n))$, then for every $n > 0$, there exist $t \geq 1$ and polynomials $f_{i,j}$, $1 \leq i \leq t$, $1 \leq j \leq m_i$, $g_{i,j}$ and $1 \leq i \leq t$, $1 \leq j \leq m_i$ in $\mathbb{Z}[x_1, \dots, x_n]$ such that:

- 1. $\text{space}(f_{i,j}) \leq s(n)$, for every $1 \leq i \leq t_1$, $1 \leq j \leq m_i$; and*
- 2. $\text{space}(g_{i,j}) \leq s(n)$, for every $1 \leq i \leq t_2$, $1 \leq j \leq m_i$; and*
- 3. $L \cap \mathbb{F}^n = \bigcup_{i=1}^t \bigcap_{j=1}^{m_i} [f_{i,j} = 0] \cap \bigcap_{j=1}^{m_i} [g_{i,j} \neq 0]$.*

Proof Sketch

- Suppose $L \in \text{space}(n^c)$ for some $c > 0$.
- Let $f_{i,j}, g_{i,j}, 1 \leq i \leq t, 1 \leq j \leq m_i$ as given in the lemma, and
- $L_n = \bigcup_{i=1}^t \bigcap_{j=1}^{m_i} [f_{i,j} = 0] \cap \bigcap j = 1^{m_i} [g_{i,j} \neq 0]$.
- Let $V_i = \bigcap_{j=1}^{m_i} [f_{i,j} = 0]$, $W_i = \bigcap_{j=1}^{m_i} [g_{i,j} \neq 0]$ and $T_i = V_i \cap W_i$.
- Then, $L_n = \bigcup_{i=1}^t T_i$. Then $L_n = \bigcup_{i=1}^t \widehat{T}_i$, where \widehat{T}_i is the Zariski closure.
- Since L_n is irreducible, $L_n = \widehat{T}_i$ for some i , i.e., $L_n \subseteq [f_{i,j} = 0]$ for some j , therefore $\text{Sym}_{n,n/2} | f_{i,j}$.

Lemma

Any polynomial divisible by $Sym_{n,n/2}$ has $2^{\Omega(n)}$ monomials.

- A contradiction to $Sym_{n,n/2} | f_{i,j}$.
- Conclusion: $L \notin Space(n^c)$ for any $c \geq 0$.

Thank You !!