

THE SNOW TEAM PROBLEM

— Clearing Directed Subgraphs by Mobile Agents —

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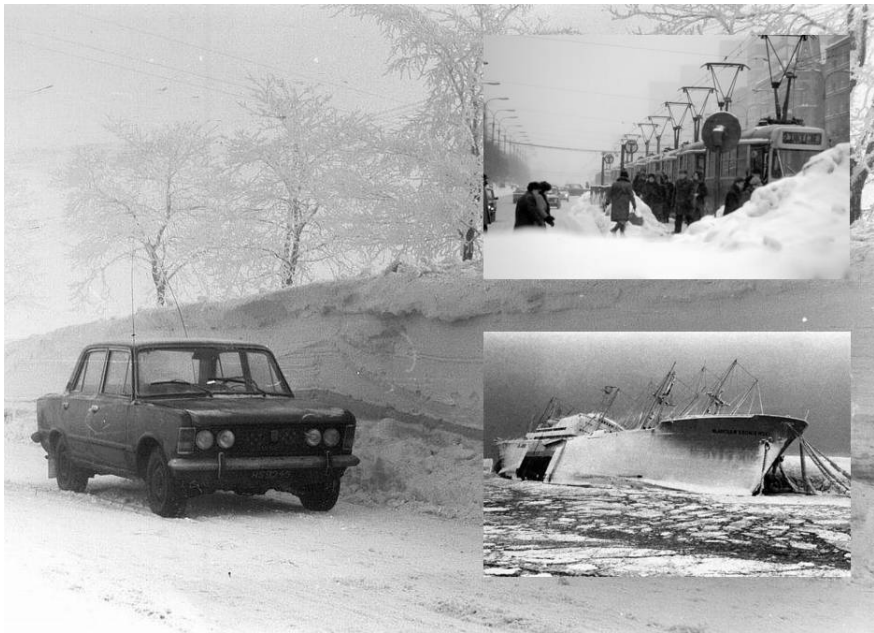


Figure: 100 years' winter in Poland in 1978.



Figure: Example of the Snow Team Problem.

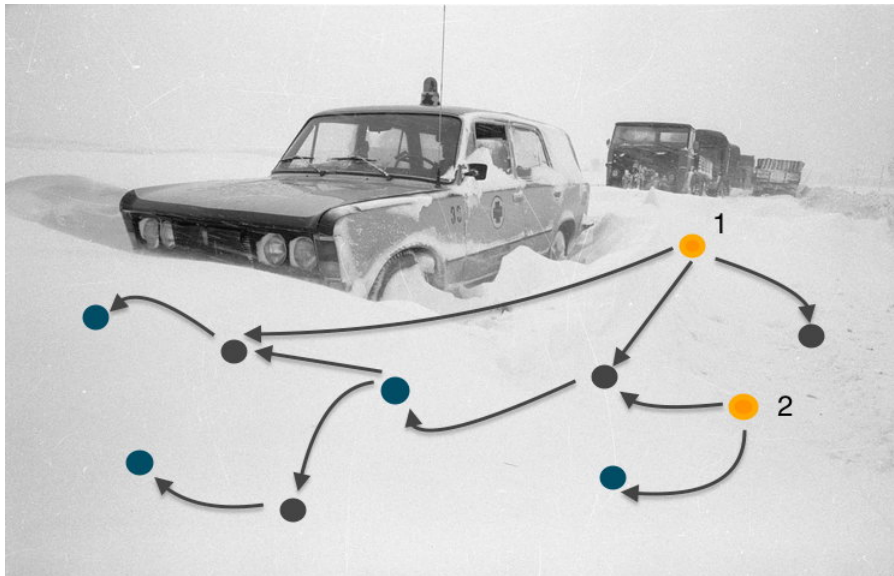


Figure: Example of the Snow Team Problem.

Introduction

Let $D = (\mathcal{V}, \mathcal{A}, F, B)$ be a vertex-weighted digraph of order n and size m , with two vertex-weight functions $F: \mathcal{V} \rightarrow \{0, 1\}$ and $B: \mathcal{V} \rightarrow \mathbb{N}$, such that its underlying graph is connected.

We denote:

- $\mathcal{F} = F^{-1}(1)$ as the locations of facilities, called *terminals*;
- $\mathcal{B} = B^{-1}(\mathbb{N}^+)$ as *snow bases*, i.e. the vertices where a (positive) number of *snow ploughs* is placed;
- $\mathbf{k}_B = \sum_{v \in \mathcal{V}} B(v)$ as the total number of snow ploughs placed in the digraph.

The Snow Team Problem

The Snow Team problem (ST)

Do there exist k_B directed walks in D , with exactly $B(v)$ starting points at each vertex $v \in \mathcal{V}$, whose edges induce a subgraph H of D such that all vertices in $F^{-1}(1)$ belong to one connected component of the underlying graph of H ?

The Snow Team Problem

Motivation

- ACT problem [Levcopoulos et al. (2014)]
- variant of the path cover problem [Beerenwinkel et al. (2015), J. Kováč (2013), Ntafos and Hakimi (1979)]
- connected graph searching problems [Best et al. (2016), Dereniowski (2012)]
- directed Steiner Tree problem [Chitnis et al. (2014), Suchý (2016), Watel et al. (2016)].

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Our Results

- ST problem is strongly NP-complete
- ST problem is FPT
- min-ST, max-ST, STU problems are FPT
- STU problem is strongly NP-complete
- ACT problem is NP-complete

NP-completeness of ST

We prove that the ST problem is NP-complete by describing a polynomial-time reduction from the Set Cover problem.

Theorem

The ST problem is strongly NP-complete even for directed acyclic graphs $D = (\mathcal{V}, \mathcal{A}, F, B)$ with $F^{-1}(1) = \mathcal{V}$ and $B(v) = 1$ if v is a source vertex in D and $B(v) = 0$ otherwise.

Definition (Tree-like solution)

Let \mathcal{W} be a set of walks (if any) that constitute a positive answer to the ST problem for a given instance $D = (\mathcal{V}, \mathcal{A}, \mathcal{F}, \mathcal{B})$. We say that \mathcal{W} is *tree-like* if all walks in \mathcal{W} are arc-distinct and the underlying graph of their union includes a Steiner tree for $\mathcal{F} \cup \mathcal{B}$. Notice that if \mathcal{W} is tree-like, then all walks in \mathcal{W} are just (simple) paths.

Tree-like Solution

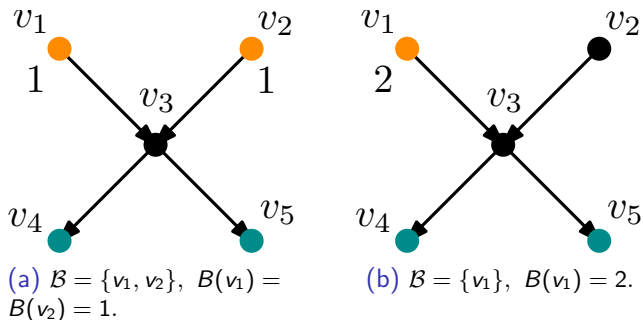


Figure: $\mathcal{F} = \{v_4, v_5\}$, orange dots denote the snow team bases and green dots denote the terminals. There exists a tree-like set of walks on (a), but not on (b).

Lemma About Transitive Closure

Lemma

A (restricted) instance $D = (\mathcal{V}, \mathcal{A}, F, B)$ admits a positive answer to the ST problem if and only if the transitive closure $\text{TC}(D) = (\mathcal{V}, \mathcal{A}', F, B)$ of D , with the same vertex-weight functions F and B , admits a positive answer to the ST problem with a tree-like set of walks whose underlying graph is of order at most $2|\mathcal{F} \cup \mathcal{B}| - 1$.

Lemma About Transitive Closure

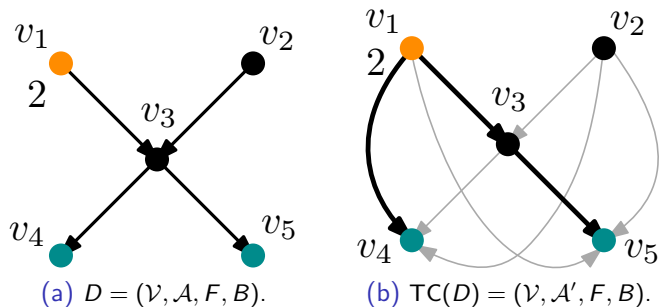


Figure: $\mathcal{F} = \{v_4, v_5\}$, $\mathcal{B} = \{v_1\}$, $B(v_1) = 2$. Solution in (b) has to be tree-like of order at most $2|\mathcal{F} \cup \mathcal{B}| - 1 = 5$.

The Tree Pattern Embedding Problem

solving ST problem for $D = (\mathcal{V}, \mathcal{A}, F, B)$



searching through tree-like solutions in $TC(D)$ with at most $2|\mathcal{F} \cup \mathcal{B}| - 1$ vertices (*tree-like-restricted ST*)



solving the problem if $TC(D)$ has a subtree $T = (\mathcal{V}_T, \mathcal{A}_T)$ of order at most $2|\mathcal{F} \cup \mathcal{B}| - 1$ such that $\mathcal{F} \cup \mathcal{B} \subseteq \mathcal{V}_T$ and all edges of T can be traversed by at most k_B snow ploughs following arc-distinct paths starting at vertices in \mathcal{B}

The Tree Pattern Embedding Problem - Example

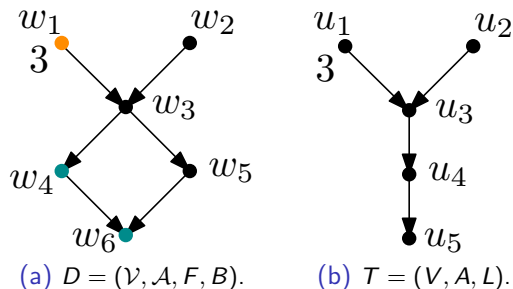


Figure: $\mathcal{F} = \{w_4, w_6\}$ and $\mathcal{B} = \{w_1\}$, $B(w_1) = 3$; $L^{-1}(0) = \{u_2, u_3, u_4, u_5\}$.

Only one solution: $u_1 \rightarrow w_1$, $u_2 \rightarrow w_2$, $u_3 \rightarrow w_3$, $u_4 \rightarrow w_4$, $u_5 \rightarrow w_6$.

The Tree Pattern Embedding Problem - Example

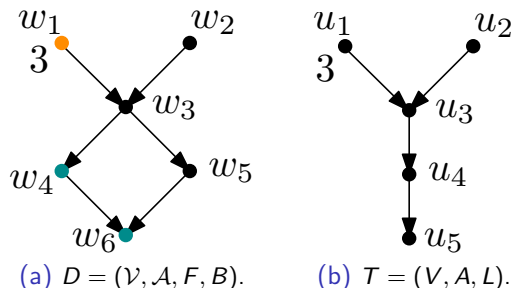


Figure: $\mathcal{F} = \{w_4, w_6\}$ and $\mathcal{B} = \{w_1\}$, $B(w_1) = 3$; $L^{-1}(0) = \{u_2, u_3, u_4, u_5\}$.

Only one solution: $u_1 \rightarrow w_1$, $u_2 \rightarrow w_2$, $u_3 \rightarrow w_3$, $u_4 \rightarrow w_4$, $u_5 \rightarrow w_6$.

We say that T has an \mathcal{S} -embedding in D , where $\mathcal{S} = \mathcal{F} \cup \mathcal{B}$.

The Tree Pattern Embedding Problem

Let $D = (\mathcal{V}, \mathcal{A}, F, B)$ be a directed graph of order n and size m , with two vertex-weight functions $F: \mathcal{V} \rightarrow \{0, 1\}$ and $B: \mathcal{V} \rightarrow \mathbb{N}$ and let $T = (V, A, L)$ be a directed vertex-weighted tree of order t , with a vertex-weight function $L: V \rightarrow \mathbb{N}$.

The Tree Pattern Embedding problem (TPE)

Does D have a subgraph $H = (\mathcal{V}_H, \mathcal{A}_H)$ isomorphic to T such that $F^{-1}(1) \cup B^{-1}(\mathbb{N}^+) \subseteq \mathcal{V}_H$ and $L(v) \leq B(h(v))$ for any vertex v of T , where h is an isomorphism of T and H ?

The TPE Problem: Polynomial Construction

monomial - a polynomial that is just a product of variables

multilinear monomial - a monomial in which each variable occurs at most once

Fact ([Koutis and Williams (2016)])

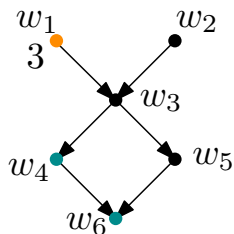
Let $P(x_1, \dots, x_n, z)$ be a polynomial represented by a monotone arithmetic circuit of size $s(n)$. There is a randomized algorithm that for the input P runs in $O^(2^k t^2 s(n))$ time and outputs “YES” with high probability if there is a monomial of the form $z^t Q(x_1, \dots, x_n)$, where $Q(x_1, \dots, x_n)$ is a multilinear monomial of degree at most k , in the sum-product expansion of P , and always outputs “NO” if there is no such monomial $z^t Q(x_1, \dots, x_n)$ in the expansion.* □

The TPE Problem: Polynomial Construction

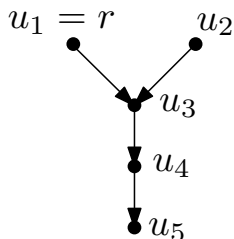
Input: digraph $D = (\mathcal{V}, \mathcal{A}, F, B)$ and directed tree $T = (V, A, L)$.

Idea: construct an appropriate polynomial $Q(X, z)$ such that $Q(X, z)$ contains a monomial of the form $z^{|\mathcal{S}|} b(X)$, where $b(X)$ is a multilinear polynomial with exactly t variables in X and $\mathcal{S} = \mathcal{F} \cup \mathcal{B}$, if and only if the t -vertex tree T has an \mathcal{S} -embedding in D .

The TPE Problem: Example



$F: D = (\mathcal{V}, \mathcal{A}, F, B)$



$F: T_1 = (\mathcal{V}_1, \mathcal{A}_1, L_1)$

$$\mathcal{F} = \{w_4, w_6\},$$

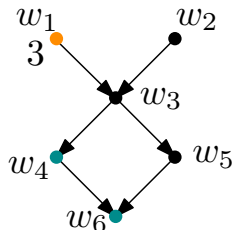
$$\mathcal{B} = \{w_1\}, B(w_1) = 3 \text{ and } L_1^{-1}(0) = V_1.$$

We calculate polynomial $Q(X, T_1)$ obtaining final result:

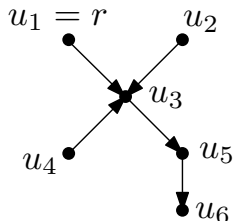
$$\begin{aligned} Q(X, T_1) &= z^2 x_{w_1} x_{w_3} x_{w_6} (z x_{w_1} + x_{w_2}) (z x_{w_4} + x_{w_5}) + \\ &\quad + z x_{w_2} x_{w_3} x_{w_6} (z x_{w_1} + x_{w_2}) (z x_{w_4} + x_{w_5}) = \\ &= \dots + 2z^3 x_{w_1} x_{w_2} x_{w_3} x_{w_4} x_{w_6}. \end{aligned}$$

It contains a monomial of the form $z^{|S|} b(X)$, which reflects that T_1 can be embedded into D .

The TPE Problem: Example



$F: D = (\mathcal{V}, \mathcal{A}, F, B)$



$F: T_2 = (V_2, A_2, L_2)$

$$\mathcal{F} = \{w_4, w_6\},$$

$$\mathcal{B} = \{w_1\}, B(w_1) = 3 \text{ and } L_2^{-1}(0) = V_2.$$

We calculate polynomial $Q(X, T_2)$ obtaining final result:

$$Q(X, T_2) = z^2 x_{w_1} x_{w_3} x_{w_6} (z x_{w_1} + x_{w_2})^2 (z x_{w_4} + x_{w_5}) + z x_{w_2} x_{w_3} x_{w_6} (z x_{w_1} + x_{w_2})^2 (z x_{w_4} + x_{w_5}).$$

It does not contain a monomial of the form $z^{|S|} b(X)$, which reflects that T_2 can not be embedded into D .

The TPE Problem: Main Theorem

Lemma

The polynomial $Q(X, z)$ contains a monomial of the form $z^{|\mathcal{S}|} b(X)$, where $b(X)$ is a multilinear polynomial with exactly t variables in X , if and only if the t -vertex tree T has an \mathcal{S} -embedding into D .

Theorem

There is a randomized algorithm that solves the TPE problem in $O^(2^t)$ time.*

The ST Problem is FPT

Theorem

There is a randomized algorithm that solves the tree-like-restricted ST problem for $D = (\mathcal{V}, \mathcal{A}, F, B)$ in $O^(144^{|\mathcal{S}|})$ time, where $\mathcal{S} = \mathcal{F} \cup \mathcal{B}$.*

Proof.

There is $O(9^{|\mathcal{S}|})$ undirected trees of order t [Otter (1948)], where $|\mathcal{S}| \leq t \leq 2|\mathcal{S}| - 1$. For each such a t -vertex candidate tree, we enumerate all orientations of its edges, in order to obtain a directed tree; there are 2^{t-1} such orientations. Therefore, we have $O(36^{|\mathcal{S}|})$ candidates for a directed oriented tree T of order t , where $|\mathcal{S}| \leq t \leq 2|\mathcal{S}| - 1$.

The ST Problem is FPT

Proof (Cont.).

For each candidate $T = (V, A)$, we establish in $O(t)$ time function L , which determines the number of (at least) needed snow ploughs at every vertex in order to traverse all arcs of T , in an arc-disjoint manner, i.e., for every vertex v is equal to $\max\{0, \deg_{\text{out}}(v) - \deg_{\text{in}}(v)\}$.

We then solve the TPE problem with the instance D and $T = (V, A, L)$ in $O^*(2^t)$ time by Theorem 6.

As already observed, if T admits a positive answer to the TPE problem for D , then D admits a positive answer to the tree-like-restricted All-ST problem. Therefore, by deciding the TPE problem for each of $O(36^{|S|})$ candidates, taking into account the independence of any two tests, we obtain a randomized algorithm for the restricted ST problem with a running time $O^*(144^{|S|})$.

The ST problem is FPT

Corollary

The ST problem admits a fixed-parameter algorithm with respect to the total number l of facilities and snow team bases, running in $2^{O(l)} \cdot \text{poly}(n)$ time, where n is the order of the input graph. □

The min-ST Problem

The Minimal Snow Team problem (ST)

For a given input n -vertex digraph $D = (\mathcal{V}, \mathcal{A}, F, B)$ what is the minimum number of snow ploughs among those available at snow team bases in $\mathcal{B} = B^{-1}(\mathbb{N}^+)$ that admits a positive answer to the (original) Snow Team problem in D ?

Corollary

The min-ST problem admits a fixed-parameter algorithm with respect to the total number l of facilities and snow team bases, running in $2^{O(l)} \cdot \text{poly}(n)$ time, where n is the order of the input graph.

The max-ST Problem

The Maximal Snow Team problem (ST)

For a given input n -vertex digraph $D = (\mathcal{V}, \mathcal{A}, F, B)$ what is the maximum number of facilities in $F^{-1}(1)$ that can be re-connected by snow ploughs located with respect to the plough-quantity function B ?

Corollary

The max-ST problem admits a fixed-parameter algorithm with respect to the total number l of facilities and snow team bases, running in $2^{O(l)} \cdot \text{poly}(n)$ time, where n is the order of the input graph.

The STU Problem

The Snow Team problem with Unspecified snow team bases (STU)

Given a weight function $F: \mathcal{V} \rightarrow \{0, 1\}$ and an integer $k \geq 1$, do there exist k directed walks in a digraph $D = (\mathcal{V}, \mathcal{A})$ whose edges induce a subgraph H of D such that the set $F^{-1}(1)$ is a subset of the vertex set of H and the underlying graph of H is connected?

Corollary

The STU problem admits a fixed-parameter algorithm with respect to the number l of facilities and the number k of snow ploughs, running in $2^{O(k+l)} \cdot \text{poly}(n)$ time, where n is the order of the input graph. □

Corollary

The STU problem is strongly NP-complete even for directed acyclic graphs $D = (\mathcal{V}, \mathcal{A}, F)$ with $F^{-1}(1) = \mathcal{V}$ and k equals the number of source vertices in D . □

Corollary

The ACT [Levcopoulos et al. (2014)] problem is NP-complete. □

Thank you for your attention!

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