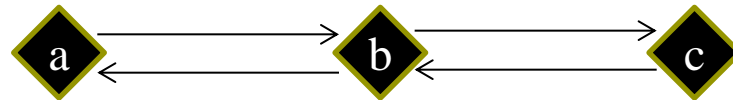


Strong Duality in Horn minimization

Endre Boros, Ondřej Čepek, Kazuhisa Makino
Charles University, Prague, Czech Republic

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Digraphs, implications, Horn 2-CNFs

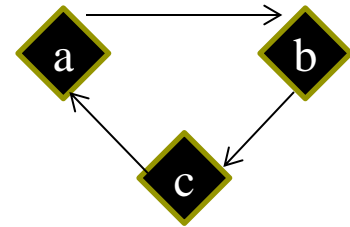
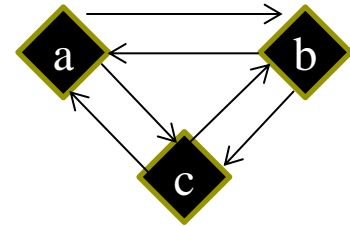


- List of arcs
 - $(a,b), (b,a), (b,c), (c,b)$
- Neighbor lists
 - $a : b$
 - $b : a,c$
 - $c : b$
- Adjacency matrix

0	1	0
1	0	1
0	1	0

Digraphs, implications, Horn 2-CNFs

- Directed paths
- Transitivity
- Transitive closure
 - Can be computed e.g. by DFS
 - Defines equivalence relation
- Transitive reduction
 - Minimum equivalent graph
 - Can be computed in polynomial time [Tarjan 74]
 - Notice: not enough to drop arcs !!



Digraphs, implications, Horn 2-CNFs

- Directed graph \leftrightarrow Set of simple implications (rules)
- $a \rightarrow b, b \rightarrow a, b \rightarrow c, c \rightarrow b$
- $a \rightarrow b, b \rightarrow ac, c \rightarrow b$
- Transitivity \leftrightarrow Logical deduction
- Transitive closure \leftrightarrow Set of all logically deducible implications (rules), implicational closure
- Transitive reduction \leftrightarrow Minimum set of implications (rules) representing the same knowledge, minimum implicational base

Digraphs, implications, Horn 2-CNFs

- Another equivalent concept: pure Horn 2-CNF

$$(\neg a \vee b) \wedge (\neg b \vee a) \wedge (\neg b \vee c) \wedge (\neg c \vee b)$$

- two quadratic clauses are resolvable if they have exactly one conflicting literal producing a resolvent
- if $C_1 = \neg a \vee x$, $C_2 = \neg x \vee b$ then $R(C_1, C_2) = \neg a \vee b$
- Transitivity, Deduction \leftrightarrow Resolution
- Transitive closure, Implicational closure \leftrightarrow Resolution closure
- Transitive reduction, Minimum base \leftrightarrow Shortest equivalent 2-CNF

Directed hypergraphs, implicational systems, pure Horn CNFs

- Generalizations of directed graphs, simple implications, and pure Horn 2-CNFs where antecedents are no longer singletons
- Directed hypergraph: $(\{a\}, b), (\{b\}, a), (\{a, c\}, d), (\{a, c\}, e)$ or $\{a\} : b, \{b\} : a, \{a, c\} : d, e$
- Implicational system: $a \rightarrow b, b \rightarrow a, ac \rightarrow d, ac \rightarrow e$ or $a \rightarrow b, b \rightarrow a, ac \rightarrow de$
- Pure Horn CNF:
 $(\neg a \vee b) \wedge (\neg b \vee a) \wedge (\neg a \vee \neg c \vee d) \wedge (\neg a \vee \neg c \vee e)$

Directed hypergraphs, implicational systems, pure Horn CNFs

- Derivation rules
 - Generalized transitivity
 - Logical deduction – Forward chaining
 - Resolution (general form)
- Complexity changes dramatically from the simple case of graphs, simple implications, and 2-CNFs
 - Closures can be of exponential size
 - Reductions are usually hard to compute

Minimal CNFs: number of clauses

- Σ_2 -complete for general CNFs [Umans 2001]
- NP-complete (Σ_1 -complete) for pure Horn CNFs [Ausiello, D'Atri, Sacca 1986]
- NP-complete for cubic pure Horn CNFs [Boros, Gruber 2012, Boros, Čepek, Kučera 2013]
- Polynomial for quadratic CNFs [folklore]
- Polynomial for acyclic and quasi-acyclic Horn CNFs [Hammer, Kogan 1995]
- Polynomial for component-wise quadr. Horn CNFs [Boros, Čepek, Kogan, Kučera 2010]
- All above results true also for total number of literals₈

Minimal CNFs: number of source sets

- Number of adjacency lists for directed hypergraphs
- Number of rules for implicational systems
- Does not make sense for general CNFs
- Polynomial for pure Horn CNFs [Maier 1980]
- Similar result was independently discovered in the implicational systems community: GD-basis [Guigues, Duquenne 1986]
- Both algorithms can be used for pure Horn CNFs

Minimal CNFs: number of clauses

Upper and lower bounds

- Given a CNF representing a function f , can we estimate the number of clauses in a minimal representation of f ?
- Upper bounds: every CNF representation of f gives an upper bound
- Lower bounds: ???
- Verification of minimality: ???

Boolean basics

- Clause C is an implicate of function f if $f \leq C$
- C is a prime implicate of f if dropping any literal means that C is no longer an implicate of f
- prime CNF, irredundant CNF
- two clauses are resolvable if they have exactly one conflicting literal producing a resolvent
- if $C_1 = A \vee x$, $C_2 = B \vee \neg x$ then $R(C_1, C_2) = A \vee B$
- $R(S)$ is a resolution closure of set S of clauses

Notation

- $P(f)$ is a set of all prime implicants of f
- $P^*(f) = \mathbf{R}(P(f))$
- Recall: completeness of resolution [Quine 1955]:
for any CNF representation $S \subseteq P^*(f)$ of a function f
we have $\mathbf{R}(S) = P^*(f)$ (and hence $P(f) \subseteq \mathbf{R}(S)$)

Essential sets of implicates: definition

- Let f be a Boolean function. Then $X \subseteq P^*(f)$ is an essential set of f if for every two resolvable clauses $C_1, C_2 \in P^*(f)$ the following implication holds:

$$R(C_1, C_2) \in X \Rightarrow C_1 \in X \text{ or } C_2 \in X$$

- Example 1: $S \subseteq P^*(f)$ s.t. $S = \mathbf{R}(S)$, $X = P^*(f) \setminus S$
(can serve as an alternative definition)
- Example 2: $t \in \{0,1\}^n$, $X(t) = \{C \in P^*(f) \mid C(t) = 0\}$
(false-point essential sets)

Essential sets of implicates: properties

- Theorem [Boros, Čepek, Kogan, Kučera 2008]:
Let $S \subseteq P^*(f)$ be arbitrary. Then S represents f if and only if $S \cap X \neq \emptyset$ for every nonempty essential set $X \subseteq P^*(f)$.
- Observation: If $P^*(f)$ contains k pair-wise disjoint nonempty essential sets then every CNF representation of f consists of at least k clauses

Essential sets of implicates: properties

- Definition: For a function f let $\text{cnf}(f)$ denote the minimum number of clauses in a CNF representation of f and $\text{ess}(f)$ the maximum number of pair-wise disjoint nonempty essential sets of f .
- Corollary (weak duality): For every function f :
$$\text{ess}(f) \leq \text{cnf}(f).$$

Essential sets: verifiable lower bounds

- Gap: There exists a cubic Horn function on 4 variables for which $\text{ess}(f) = 4$ and $\text{cnf}(f) = 5$.
- More on the size of this gap can be found in [Hellerstein, Kletenik 2011]
- Definition: For a function f let $\text{ess}^*(f)$ denote the maximum number of vectors \mathbf{t} such that $\mathbf{X}(\mathbf{t})$'s are pairwise disjoint nonempty essential sets of f .
- Theorem: [Čepek, Kučera, Savický 2009]
For every function f : $\text{ess}^*(f) = \text{ess}(f)$.
- $\text{cnf}(f) \geq \text{ess}^*(f) \geq k$ can be verified by listing k vectors (falsepoints of f) and checking disjointness of $\mathbf{X}(\mathbf{t})$ 's¹⁶

Body disjoint essential sets

- Definition: Let f be a pure Horn function. Two essential sets $X, Y \subseteq P^*(f)$ are called body disjoint if there is no pair of clauses $C \in X$ and $C' \in Y$ such that C and C' have the same body (source set).
- Definition: For a pure Horn function f let $\text{body}(f)$ denote the minimum number of bodies (source sets) in a CNF representation of f and let $\text{bess}(f)$ denote the maximum number of pair-wise body disjoint nonempty essential sets of f .
- Weak duality: For every pure Horn function f :
$$\text{bess}(f) \leq \text{body}(f).$$

Body disjoint essential sets

- Theorem (strong duality): [Boros, Čepek, Makino]
 $\text{bess}(f) = \text{body}(f)$.
- Again it is sufficient to consider falsepoints: for any two falsepoints the body-disjointness of their essential sets can be tested in polynomial time.
- This result in some sense explains why the number of bodies (source sets) is the only „measure“ for which pure Horn minimization is poly-time solvable.

Algorithmic consequences

- Strong duality implies a conceptually very simple minimization algorithm
- 1. Right saturation (use forward chaining on every source set to add all logical consequences to the right hand side of the list)
- 2. Drop redundant lists
- Overall (asymptotic) complexity same as previous algorithms, the output is not unique, but fewer steps are required.



Thank you.