

Undecidability of the Lambek Calculus Extended with Subexponential and Bracket Modalities

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Why This Talk?

Why This Talk?



Why This Talk?

A study on the borderline of **practical motivation**
and **undecidability**.

Linguistic Intro: Lambek Grammar

John loves Mary

Linguistic Intro: Lambek Grammar

John	loves	Mary
N	$(N \setminus S) / N$	N

Linguistic Intro: Lambek Grammar

John loves Mary
 N $(N \setminus S) / N$ N $\rightarrow S$

Linguistic Intro: Lambek Grammar

\vdash John loves Mary
 N $(N \setminus S) / N$ N $\rightarrow S$

Linguistic Intro: Lambek Grammar

\vdash John loves Mary
 N $(N \setminus S) / N$ N $\rightarrow S$

 the girl whom John loves

Linguistic Intro: Lambek Grammar

John loves Mary
⊢ N $(N \setminus S) / N$ N $\rightarrow S$

the girl whom John loves
 N / CN CN $(CN \setminus CN) / (S / N)$ N $(N \setminus S) / N$

Linguistic Intro: Lambek Grammar

John loves Mary
⊢ N $(N \setminus S) / N$ N $\rightarrow S$

the girl whom
 N / CN CN $(CN \setminus CN) / (S / N)$

John loves
 N $(N \setminus S) / N$
⏟
 $\rightarrow S / N$

Linguistic Intro: Lambek Grammar

John loves Mary
 $\vdash N \quad (N \setminus S) / N \quad N \rightarrow S$

the girl whom John loves
 $\vdash N / CN \quad CN \quad (CN \setminus CN) / (S / N) \quad N \quad (N \setminus S) / N \rightarrow N$

$\underbrace{\hspace{15em}}_{\rightarrow S / N}$

the boy who loves Mary

Linguistic Intro: Lambek Grammar

John loves Mary
 $\vdash N \quad (N \setminus S) / N \quad N \rightarrow S$

the girl whom John loves
 $\vdash N / CN \quad CN \quad (CN \setminus CN) / (S / N) \quad N \quad (N \setminus S) / N \rightarrow N$
 $\underbrace{\hspace{15em}}_{\rightarrow S / N}$

the boy who loves Mary
 $N / CN \quad CN \quad (CN \setminus CN) / (N \setminus S) \quad (N \setminus S) / N \quad N$

Linguistic Intro: Lambek Grammar

John loves Mary
 $\vdash N \quad (N \setminus S) / N \quad N \rightarrow S$

the girl whom John loves
 $\vdash N / CN \quad CN \quad (CN \setminus CN) / (S / N) \quad N \quad (N \setminus S) / N \rightarrow N$
 $\underbrace{\hspace{15em}}_{\rightarrow S / N}$

the boy who loves Mary
 $N / CN \quad CN \quad (CN \setminus CN) / (N \setminus S) \quad (N \setminus S) / N \quad N$
 $\underbrace{\hspace{15em}}_{\rightarrow N \setminus S}$

Linguistic Intro: Lambek Grammar

\vdash John loves Mary
 $N \quad (N \setminus S) / N \quad N \rightarrow S$

\vdash the girl whom John loves
 $N / CN \quad CN \quad (CN \setminus CN) / (S / N) \quad N \quad (N \setminus S) / N \rightarrow N$
 $\underbrace{\hspace{10em}}_{\rightarrow S / N}$

\vdash the boy who loves Mary
 $N / CN \quad CN \quad (CN \setminus CN) / (N \setminus S) \quad (N \setminus S) / N \quad N \rightarrow N$
 $\underbrace{\hspace{10em}}_{\rightarrow N \setminus S}$

Linguistic Intro: Lambek Grammar

⊢ John loves Mary
⊢ $N \quad (N \setminus S) / N \quad N \rightarrow S$

⊢ the girl whom_i John loves e_i
⊢ $N / CN \quad CN \quad (CN \setminus CN) / (S / N) \quad N \quad (N \setminus S) / N \rightarrow N$
 $\underbrace{\hspace{15em}}_{\rightarrow S / N}$

⊢ the boy who_i e_i loves Mary
⊢ $N / CN \quad CN \quad (CN \setminus CN) / (N \setminus S) \quad (N \setminus S) / N \quad N \rightarrow N$
 $\underbrace{\hspace{15em}}_{\rightarrow N \setminus S}$

The Lambek Calculus

[Lambek 1958, 1961]

$$\overline{A \rightarrow A}$$

$$\frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, B / A, \Pi, \Delta_2 \rightarrow C} (/ \rightarrow) \quad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} (\rightarrow /)$$

$$\frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \rightarrow C} (\setminus \rightarrow) \quad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} (\rightarrow \setminus)$$

$$\frac{\Delta_1, A, B, \Delta_2 \rightarrow C}{\Delta_1, A \cdot B, \Delta_2 \rightarrow C} (\cdot \rightarrow) \quad \frac{\Pi_1 \rightarrow A \quad \Pi_2 \rightarrow B}{\Pi_1, \Pi_2 \rightarrow A \cdot B} (\rightarrow \cdot)$$

$$\frac{\Pi \rightarrow A \quad \Gamma, A, \Delta \rightarrow C}{\Gamma, \Pi, \Delta \rightarrow C} (\text{cut})$$

The Lambek Calculus

[Lambek 1958, 1961]

$$\overline{A \rightarrow A}$$

$$\frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, B / A, \Pi, \Delta_2 \rightarrow C} (/ \rightarrow) \quad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} (\rightarrow /)$$

$$\frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \rightarrow C} (\setminus \rightarrow) \quad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} (\rightarrow \setminus)$$

$$\frac{\Delta_1, A, B, \Delta_2 \rightarrow C}{\Delta_1, A \cdot B, \Delta_2 \rightarrow C} (\cdot \rightarrow) \quad \frac{\Pi_1 \rightarrow A \quad \Pi_2 \rightarrow B}{\Pi_1, \Pi_2 \rightarrow A \cdot B} (\rightarrow \cdot)$$

$$\frac{\Pi \rightarrow A \quad \Gamma, A, \Delta \rightarrow C}{\Gamma, \Pi, \Delta \rightarrow C} (\text{cut})$$

- ▶ Notice: we don't impose Lambek's non-emptiness restriction on the left-hand sides of sequents.

The Lambek Calculus

[Lambek 1958, 1961]

$$\overline{A \rightarrow A}$$

$$\frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, B / A, \Pi, \Delta_2 \rightarrow C} (/ \rightarrow) \quad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} (\rightarrow /)$$

$$\frac{\Pi \rightarrow A \quad \Delta_1, B, \Delta_2 \rightarrow C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \rightarrow C} (\setminus \rightarrow) \quad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} (\rightarrow \setminus)$$

$$\frac{\Delta_1, A, B, \Delta_2 \rightarrow C}{\Delta_1, A \cdot B, \Delta_2 \rightarrow C} (\cdot \rightarrow) \quad \frac{\Pi_1 \rightarrow A \quad \Pi_2 \rightarrow B}{\Pi_1, \Pi_2 \rightarrow A \cdot B} (\rightarrow \cdot)$$

$$\frac{\Pi \rightarrow A \quad \Gamma, A, \Delta \rightarrow C}{\Gamma, \Pi, \Delta \rightarrow C} (\text{cut})$$

- ▶ Notice: we don't impose Lambek's non-emptiness restriction on the left-hand sides of sequents.
- ▶ The cut rule is eliminable [Lambek 1958].

Unwanted Derivations

book

which

John laughed without reading

Unwanted Derivations

book which John laughed without reading
 CN $(CN \setminus CN) / (S / N)$ $\underbrace{\hspace{15em}}_{S / N}$

Unwanted Derivations

\vdash book CN which $(CN \setminus CN) / (S / N)$ $\underbrace{\text{John laughed without reading}}_{S / N} \rightarrow CN$

Unwanted Derivations

\vdash book which John laughed without reading \rightarrow CN
 CN (CN \ CN) / (S / N) $\underbrace{\hspace{10em}}_{S / N}$

Unwanted Derivations

* book which John laughed without reading
⊢ CN (CN \ CN) / (S / N) $\underbrace{\hspace{10em}}_{S / N}$ → CN

Unwanted Derivations

* book which John laughed without reading
⊢ CN $(CN \setminus CN) / (S / N)$ $\underbrace{\hspace{10em}}_{S / N}$ $\rightarrow CN$

* girl who John likes Mary and Pete likes

Unwanted Derivations

* book which John laughed without reading
⊢ CN (CN \ CN) / (S / N) $\underbrace{\hspace{15em}}_{S / N}$ → CN

* girl who John likes Mary and Pete likes
⊢ CN (CN \ CN) / (S / N) $\underbrace{\hspace{15em}}_{S / N}$ → CN

Unwanted Derivations

* book which John laughed without reading
⊢ CN $(CN \setminus CN) / (S / N)$ $\underbrace{\hspace{15em}}_{S / N}$ $\rightarrow CN$

* girl who John likes Mary and Pete likes
⊢ CN $(CN \setminus CN) / (S / N)$ $\underbrace{\hspace{15em}}_{S / N}$ $\rightarrow CN$

(cf. “John likes Mary and Pete likes Kate” $\rightarrow S$; “and” is of type $S \setminus S / S$)

The Lambek Calculus with Brackets

[Morrill 1992, Moortgat 1995]

$$\begin{array}{c}
 \overline{A \rightarrow A} \\
 \\
 \frac{\Pi \rightarrow A \quad \Delta(B) \rightarrow C}{\Delta(\Pi, A \setminus B) \rightarrow C} \quad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} \quad \frac{\Gamma(A, B) \rightarrow C}{\Gamma(A \cdot B) \rightarrow C} \\
 \\
 \frac{\Pi \rightarrow A \quad \Delta(B) \rightarrow C}{\Delta(B / A, \Pi) \rightarrow C} \quad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} \quad \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \cdot B} \\
 \\
 \frac{\Delta([A]) \rightarrow C}{\Delta(\langle \rangle A) \rightarrow C} \quad \frac{\Pi \rightarrow A}{[\Pi] \rightarrow \langle \rangle A} \quad \frac{\Delta(A) \rightarrow C}{\Delta([\]^{-1} A) \rightarrow C} \quad \frac{[\Pi] \rightarrow A}{\Pi \rightarrow [\]^{-1} A}
 \end{array}$$

The Lambek Calculus with Brackets

[Morrill 1992, Moortgat 1995]

$$\begin{array}{c}
 \overline{A \rightarrow A} \\
 \\
 \frac{\Pi \rightarrow A \quad \Delta(B) \rightarrow C}{\Delta(\Pi, A \setminus B) \rightarrow C} \quad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} \quad \frac{\Gamma(A, B) \rightarrow C}{\Gamma(A \cdot B) \rightarrow C} \\
 \\
 \frac{\Pi \rightarrow A \quad \Delta(B) \rightarrow C}{\Delta(B / A, \Pi) \rightarrow C} \quad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} \quad \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \cdot B} \\
 \\
 \frac{\Delta([A]) \rightarrow C}{\Delta(\langle \rangle A) \rightarrow C} \quad \frac{\Pi \rightarrow A}{[\Pi] \rightarrow \langle \rangle A} \quad \frac{\Delta(A) \rightarrow C}{\Delta([\]^{-1} A) \rightarrow C} \quad \frac{[\Pi] \rightarrow A}{\Pi \rightarrow [\]^{-1} A}
 \end{array}$$

- Brackets introduce *controlled non-associativity*.

The Lambek Calculus with Brackets

[Morrill 1992, Moortgat 1995]

$$\begin{array}{c}
 \overline{A \rightarrow A} \\
 \\
 \frac{\Pi \rightarrow A \quad \Delta(B) \rightarrow C}{\Delta(\Pi, A \setminus B) \rightarrow C} \quad \frac{A, \Pi \rightarrow B}{\Pi \rightarrow A \setminus B} \quad \frac{\Gamma(A, B) \rightarrow C}{\Gamma(A \cdot B) \rightarrow C} \\
 \\
 \frac{\Pi \rightarrow A \quad \Delta(B) \rightarrow C}{\Delta(B / A, \Pi) \rightarrow C} \quad \frac{\Pi, A \rightarrow B}{\Pi \rightarrow B / A} \quad \frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \cdot B} \\
 \\
 \frac{\Delta([A]) \rightarrow C}{\Delta(\langle \rangle A) \rightarrow C} \quad \frac{\Pi \rightarrow A}{[\Pi] \rightarrow \langle \rangle A} \quad \frac{\Delta(A) \rightarrow C}{\Delta([\]^{-1} A) \rightarrow C} \quad \frac{[\Pi] \rightarrow A}{\Pi \rightarrow [\]^{-1} A}
 \end{array}$$

- ▶ Brackets introduce *controlled non-associativity*.
- ▶ Cut elimination proved by Moortgat [1996].

Islands: Blocking Unwanted Derivations Using Brackets

Islands: Blocking Unwanted Derivations Using Brackets

- ▶ book which John laughed without reading

Islands: Blocking Unwanted Derivations Using Brackets

- ▶ book which John laughed [without reading]

Islands: Blocking Unwanted Derivations Using Brackets

- ▶ book which John laughed [without reading]

$CN, (CN \setminus CN) / (S / CN), N, N \setminus S, [[]^{-1}((N \setminus S) \setminus (N \setminus S)) / (N \setminus S), (N \setminus S) / N] \rightarrow CN$

Islands: Blocking Unwanted Derivations Using Brackets

- ▶ book which John laughed [without reading]

$CN, (CN \setminus CN) / (S / CN), N, N \setminus S, []^{-1}((N \setminus S) \setminus (N \setminus S)) / (N \setminus S), (N \setminus S) / N] \rightarrow CN$

This sequent is not derivable.

Islands: Blocking Unwanted Derivations Using Brackets

- ▶ book which John laughed [without reading]

$CN, (CN \setminus CN) / (S / CN), N, N \setminus S, [[]^{-1}((N \setminus S) \setminus (N \setminus S)) / (N \setminus S), (N \setminus S) / N] \rightarrow CN$

This sequent is not derivable.

- ▶ girl who John likes Mary and Pete likes

Islands: Blocking Unwanted Derivations Using Brackets

- ▶ book which John laughed [without reading]

$CN, (CN \setminus CN) / (S / CN), N, N \setminus S, []^{-1}((N \setminus S) \setminus (N \setminus S)) / (N \setminus S), (N \setminus S) / N] \rightarrow CN$

This sequent is not derivable.

- ▶ girl who [John likes Mary and Pete likes]

Islands: Blocking Unwanted Derivations Using Brackets

- ▶ book which John laughed [without reading]

$CN, (CN \setminus CN) / (S / CN), N, N \setminus S, [\]^{-1}((N \setminus S) \setminus (N \setminus S)) / (N \setminus S), (N \setminus S) / N] \rightarrow CN$

This sequent is not derivable.

- ▶ girl who [John likes Mary and Pete likes]

$CN, (CN \setminus CN) / (S / CN), [N, (N \setminus S) / N, N, (S \setminus [\]^{-1}S) / S, N, (N \setminus S) / N] \rightarrow CN$

Islands: Blocking Unwanted Derivations Using Brackets

- ▶ book which John laughed [without reading]

$CN, (CN \setminus CN) / (S / CN), N, N \setminus S, [\]^{-1}((N \setminus S) \setminus (N \setminus S)) / (N \setminus S), (N \setminus S) / N] \rightarrow CN$

This sequent is not derivable.

- ▶ girl who [John likes Mary and Pete likes]

$CN, (CN \setminus CN) / (S / CN), [N, (N \setminus S) / N, N, (S \setminus [\]^{-1}S) / S, N, (N \setminus S) / N] \rightarrow CN$

Neither is this one.

Subexponential: Medial Extraction

the girl

whom

John met

yesterday

Subexponential: Medial Extraction

the girl

whom_{*i*}

John met *e_i* yesterday

Subexponential: Medial Extraction

the girl

whom_{*i*}

John met *e_i* yesterday



Subexponential: Medial Extraction

the girl

whom_{*i*}

John met e_i yesterday

$\underbrace{\hspace{10em}}_{\rightarrow S / !N}$

$$\frac{\Delta(!A, \Gamma) \rightarrow C}{\Delta(\Gamma, !A) \rightarrow C} \text{ (perm}_1\text{)}$$

$$\frac{\Delta(A) \rightarrow C}{\Delta(!A) \rightarrow C} (! \rightarrow)$$

Subexponential: Medial Extraction

the girl

whom_{*i*}

John met *e_i* yesterday

$$\underbrace{\hspace{10em}}_{\rightarrow S / !N}$$

$$\frac{\Delta(!A, \Gamma) \rightarrow C}{\Delta(\Gamma, !A) \rightarrow C} (\text{perm}_1) \quad \frac{\Delta(A) \rightarrow C}{\Delta(!A) \rightarrow C} (! \rightarrow)$$

$$\frac{\frac{N, (N \setminus S) / N, N, (N \setminus S) \setminus (N \setminus S) \rightarrow S}{N, (N \setminus S) / N, !N, (N \setminus S) \setminus (N \setminus S) \rightarrow S} (! \rightarrow)}{\frac{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S), !N \rightarrow S}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S) \rightarrow S / !N} (\text{perm}_1)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S) \rightarrow S / !N} (\rightarrow /)$$

Subexponential: Medial Extraction

the girl whom_i John met e_i yesterday

$(CN \setminus CN) / (S / !N)$ $\underbrace{\hspace{10em}}_{\rightarrow S / !N}$

$$\frac{\Delta(!A, \Gamma) \rightarrow C}{\Delta(\Gamma, !A) \rightarrow C} (\text{perm}_1) \qquad \frac{\Delta(A) \rightarrow C}{\Delta(!A) \rightarrow C} (! \rightarrow)$$

$$\frac{\frac{N, (N \setminus S) / N, N, (N \setminus S) \setminus (N \setminus S) \rightarrow S}{N, (N \setminus S) / N, !N, (N \setminus S) \setminus (N \setminus S) \rightarrow S} (! \rightarrow)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S), !N \rightarrow S} (\text{perm}_1)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S) \rightarrow S / !N} (\rightarrow /)$$

Subexponential: Medial Extraction

the girl whom_i John met e_i yesterday
 ... $(CN \setminus CN) / (S / !N)$ $\underbrace{\hspace{10em}}_{\rightarrow S / !N}$ $\rightarrow N$

$$\frac{\Delta(!A, \Gamma) \rightarrow C}{\Delta(\Gamma, !A) \rightarrow C} (\text{perm}_1) \qquad \frac{\Delta(A) \rightarrow C}{\Delta(!A) \rightarrow C} (! \rightarrow)$$

$$\frac{\frac{N, (N \setminus S) / N, N, (N \setminus S) \setminus (N \setminus S) \rightarrow S}{N, (N \setminus S) / N, !N, (N \setminus S) \setminus (N \setminus S) \rightarrow S} (! \rightarrow)}{\frac{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S), !N \rightarrow S}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S) \rightarrow S / !N} (\text{perm}_1)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S) \rightarrow S / !N} (\rightarrow /)$$

Subexponential: Parasitic Extraction

the paper that John signed without reading

Subexponential: Parasitic Extraction

the paper that_{*j*} John signed *e_j* without reading *e_j*

Subexponential: Parasitic Extraction

the paper that_{*j*} $\underbrace{\text{John signed } e_j \text{ without reading } e_j}_{\rightarrow S / !N}$

Subexponential: Parasitic Extraction

the paper that_{*j*} John signed *e_j* without reading *e_j*

$\underbrace{\hspace{15em}}_{\rightarrow S / !N}$

$$\frac{\Delta(!A, \Gamma) \rightarrow C}{\Delta(\Gamma, !A) \rightarrow C} \text{ (perm}_1\text{)} \quad \frac{\Delta(A) \rightarrow C}{\Delta(!A) \rightarrow C} (! \rightarrow)$$

$$\frac{\Delta(!A, !A) \rightarrow C}{\Delta(!A) \rightarrow C} \text{ (contr)}$$

Subexponential: Parasitic Extraction

the paper that_{*j*} John signed *e_i* [without reading *e_i*]

$\underbrace{\hspace{15em}}_{\rightarrow S / !N}$

$$\frac{\Delta(!A, \Gamma) \rightarrow C}{\Delta(\Gamma, !A) \rightarrow C} \text{ (perm}_1\text{)} \quad \frac{\Delta(A) \rightarrow C}{\Delta(!A) \rightarrow C} (! \rightarrow)$$

$$\frac{\Delta(!A, !A) \rightarrow C}{\Delta(!A) \rightarrow C} \text{ (contr)}$$

Subexponential: Parasitic Extraction

the paper that_{*j*} John signed *e_i* [without reading *e_i*]

$\underbrace{\hspace{15em}}_{\rightarrow S / !N}$

$$\frac{\Delta(!A, \Gamma) \rightarrow C}{\Delta(\Gamma, !A) \rightarrow C} \text{ (perm}_1\text{)} \qquad \frac{\Delta(A) \rightarrow C}{\Delta(!A) \rightarrow C} (! \rightarrow)$$

$$\frac{\Delta(!A_1, \dots, !A_n, [!A_1, \dots, !A_n, \Gamma]) \rightarrow B}{\Delta(!A_1, \dots, !A_n, \Gamma) \rightarrow B} \text{ (contr}_b\text{)}$$

Subexponential: Parasitic Extraction

the paper that_{*j*} John signed *e_i* [without reading *e_i*]

$\underbrace{\hspace{15em}}_{\rightarrow S / !N}$

$$\frac{\Delta(!A, \Gamma) \rightarrow C}{\Delta(\Gamma, !A) \rightarrow C} \text{ (perm}_1\text{)} \qquad \frac{\Delta(A) \rightarrow C}{\Delta(!A) \rightarrow C} (! \rightarrow)$$

$\frac{\Delta(!A_1, \dots, !A_n, [!A_1, \dots, !A_n, \Gamma]) \rightarrow B}{\Delta(!A_1, \dots, !A_n, \Gamma) \rightarrow B} \text{ (contr}_b\text{)}$	causes undecidability
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The Lambek Calculus with Subexponential and Bracket Modalities ($!_b \mathbf{L}^1$)

$$\begin{array}{c}
 \overline{A \rightarrow A} \quad \overline{\Lambda \rightarrow \mathbf{1}} \\
 \frac{\Gamma \rightarrow B \quad \Delta(C) \rightarrow D}{\Delta(C / B, \Gamma) \rightarrow D} (\ / \rightarrow) \quad \frac{\Gamma, B \rightarrow C}{\Gamma \rightarrow C / B} (\rightarrow /) \quad \frac{\Delta(A, B) \rightarrow D}{\Delta(A \cdot B) \rightarrow D} (\cdot \rightarrow) \\
 \frac{\Gamma \rightarrow A \quad \Delta(C) \rightarrow D}{\Delta(\Gamma, A \setminus C) \rightarrow D} (\ \setminus \rightarrow) \quad \frac{A, \Gamma \rightarrow C}{\Gamma \rightarrow A \setminus C} (\rightarrow \setminus) \quad \frac{\Gamma_1 \rightarrow A \quad \Gamma_2 \rightarrow B}{\Gamma_1, \Gamma_2 \rightarrow A \cdot B} (\rightarrow \cdot) \\
 \frac{\Delta(\Lambda) \rightarrow A}{\Delta(\mathbf{1}) \rightarrow A} (\mathbf{1} \rightarrow) \quad \frac{\Delta([A]) \rightarrow C}{\Delta(\langle \rangle A) \rightarrow C} (\langle \rangle \rightarrow) \quad \frac{\Pi \rightarrow A}{[\Pi] \rightarrow \langle \rangle A} (\rightarrow \langle \rangle) \\
 \frac{\Gamma(A) \rightarrow B}{\Gamma(!A) \rightarrow B} (! \rightarrow) \quad \frac{\Delta(A) \rightarrow C}{\Delta([\]^{-1} A) \rightarrow C} ([]^{-1} \rightarrow) \quad \frac{[\Pi] \rightarrow A}{\Pi \rightarrow []^{-1} A} (\rightarrow []^{-1}) \\
 \frac{!A_1, \dots, !A_n \rightarrow A}{!A_1, \dots, !A_n \rightarrow !A} (\rightarrow !) \quad \frac{\Delta(!A_1, \dots, !A_n, [!A_1, \dots, !A_n, \Gamma]) \rightarrow B}{\Delta(!A_1, \dots, !A_n, \Gamma) \rightarrow B} (\text{contr}_b) \\
 \frac{\Delta(!A, \Gamma) \rightarrow B}{\Delta(\Gamma, !A) \rightarrow B} (\text{perm}_1) \quad \frac{\Delta(\Gamma, !A) \rightarrow B}{\Delta(!A, \Gamma) \rightarrow B} (\text{perm}_2) \quad \frac{\Pi \rightarrow A \quad \Delta(A) \rightarrow C}{\Delta(\Pi) \rightarrow C} (\text{cut})
 \end{array}$$

The Lambek Calculus with Subexponential and Bracket Modalities ($!_b \mathbf{L}^1$)

$$\begin{array}{c}
 \overline{A \rightarrow A} \quad \overline{\Lambda \rightarrow \mathbf{1}} \\
 \frac{\Gamma \rightarrow B \quad \Delta(C) \rightarrow D}{\Delta(C / B, \Gamma) \rightarrow D} (\diagdown \rightarrow) \quad \frac{\Gamma, B \rightarrow C}{\Gamma \rightarrow C / B} (\rightarrow \diagdown) \quad \frac{\Delta(A, B) \rightarrow D}{\Delta(A \cdot B) \rightarrow D} (\cdot \rightarrow) \\
 \frac{\Gamma \rightarrow A \quad \Delta(C) \rightarrow D}{\Delta(\Gamma, A \setminus C) \rightarrow D} (\diagup \rightarrow) \quad \frac{A, \Gamma \rightarrow C}{\Gamma \rightarrow A \setminus C} (\rightarrow \diagup) \quad \frac{\Gamma_1 \rightarrow A \quad \Gamma_2 \rightarrow B}{\Gamma_1, \Gamma_2 \rightarrow A \cdot B} (\rightarrow \cdot) \\
 \frac{\Delta(\Lambda) \rightarrow A}{\Delta(\mathbf{1}) \rightarrow A} (\mathbf{1} \rightarrow) \quad \frac{\Delta([A]) \rightarrow C}{\Delta(\langle \rangle A) \rightarrow C} (\langle \rangle \rightarrow) \quad \frac{\Pi \rightarrow A}{[\Pi] \rightarrow \langle \rangle A} (\rightarrow \langle \rangle) \\
 \frac{\Gamma(A) \rightarrow B}{\Gamma(!A) \rightarrow B} (! \rightarrow) \quad \frac{\Delta(A) \rightarrow C}{\Delta([\square]^{-1}A) \rightarrow C} ([\square]^{-1} \rightarrow) \quad \frac{[\Pi] \rightarrow A}{\Pi \rightarrow [\square]^{-1}A} (\rightarrow [\square]^{-1}) \\
 \frac{!A_1, \dots, !A_n \rightarrow A}{!A_1, \dots, !A_n \rightarrow !A} (\rightarrow !) \quad \frac{\Delta(!A_1, \dots, !A_n, [!A_1, \dots, !A_n, \Gamma]) \rightarrow B}{\Delta(!A_1, \dots, !A_n, \Gamma) \rightarrow B} (\text{contr}_{\mathbf{b}}) \\
 \frac{\Delta(!A, \Gamma) \rightarrow B}{\Delta(\Gamma, !A) \rightarrow B} (\text{perm}_1) \quad \frac{\Delta(\Gamma, !A) \rightarrow B}{\Delta(!A, \Gamma) \rightarrow B} (\text{perm}_2) \quad \frac{\Pi \rightarrow A \quad \Delta(A) \rightarrow C}{\Delta(\Pi) \rightarrow C} (\text{cut})
 \end{array}$$

► A fragment of $\mathbf{Db!}_b$ by Morrill and Valentín, 2015.

The Lambek Calculus with Subexponential and Bracket Modalities ($!_b \mathbf{L}^1$)

$$\begin{array}{c}
 \overline{A \rightarrow A} \quad \overline{\Lambda \rightarrow \mathbf{1}} \\
 \\
 \frac{\Gamma \rightarrow B \quad \Delta(C) \rightarrow D}{\Delta(C / B, \Gamma) \rightarrow D} (\diagdown \rightarrow) \quad \frac{\Gamma, B \rightarrow C}{\Gamma \rightarrow C / B} (\rightarrow \diagdown) \quad \frac{\Delta(A, B) \rightarrow D}{\Delta(A \cdot B) \rightarrow D} (\cdot \rightarrow) \\
 \\
 \frac{\Gamma \rightarrow A \quad \Delta(C) \rightarrow D}{\Delta(\Gamma, A \setminus C) \rightarrow D} (\setminus \rightarrow) \quad \frac{A, \Gamma \rightarrow C}{\Gamma \rightarrow A \setminus C} (\rightarrow \setminus) \quad \frac{\Gamma_1 \rightarrow A \quad \Gamma_2 \rightarrow B}{\Gamma_1, \Gamma_2 \rightarrow A \cdot B} (\rightarrow \cdot) \\
 \\
 \frac{\Delta(\Lambda) \rightarrow A}{\Delta(\mathbf{1}) \rightarrow A} (\mathbf{1} \rightarrow) \quad \frac{\Delta([A]) \rightarrow C}{\Delta(\langle \rangle A) \rightarrow C} (\langle \rangle \rightarrow) \quad \frac{\Pi \rightarrow A}{[\Pi] \rightarrow \langle \rangle A} (\rightarrow \langle \rangle) \\
 \\
 \frac{\Gamma(A) \rightarrow B}{\Gamma(!A) \rightarrow B} (! \rightarrow) \quad \frac{\Delta(A) \rightarrow C}{\Delta([\square]^{-1}A) \rightarrow C} ([\square]^{-1} \rightarrow) \quad \frac{[\Pi] \rightarrow A}{\Pi \rightarrow [\square]^{-1}A} (\rightarrow [\square]^{-1}) \\
 \\
 \frac{!A_1, \dots, !A_n \rightarrow A}{!A_1, \dots, !A_n \rightarrow !A} (\rightarrow !) \quad \frac{\Delta(!A_1, \dots, !A_n, [!A_1, \dots, !A_n, \Gamma]) \rightarrow B}{\Delta(!A_1, \dots, !A_n, \Gamma) \rightarrow B} (\text{contr}_{\mathbf{b}}) \\
 \\
 \frac{\Delta(!A, \Gamma) \rightarrow B}{\Delta(\Gamma, !A) \rightarrow B} (\text{perm}_1) \quad \frac{\Delta(\Gamma, !A) \rightarrow B}{\Delta(!A, \Gamma) \rightarrow B} (\text{perm}_2) \quad \frac{\Pi \rightarrow A \quad \Delta(A) \rightarrow C}{\Delta(\Pi) \rightarrow C} (\text{cut})
 \end{array}$$

► A fragment of $\mathbf{Db!}_b$ by Morrill and Valentín, 2015.

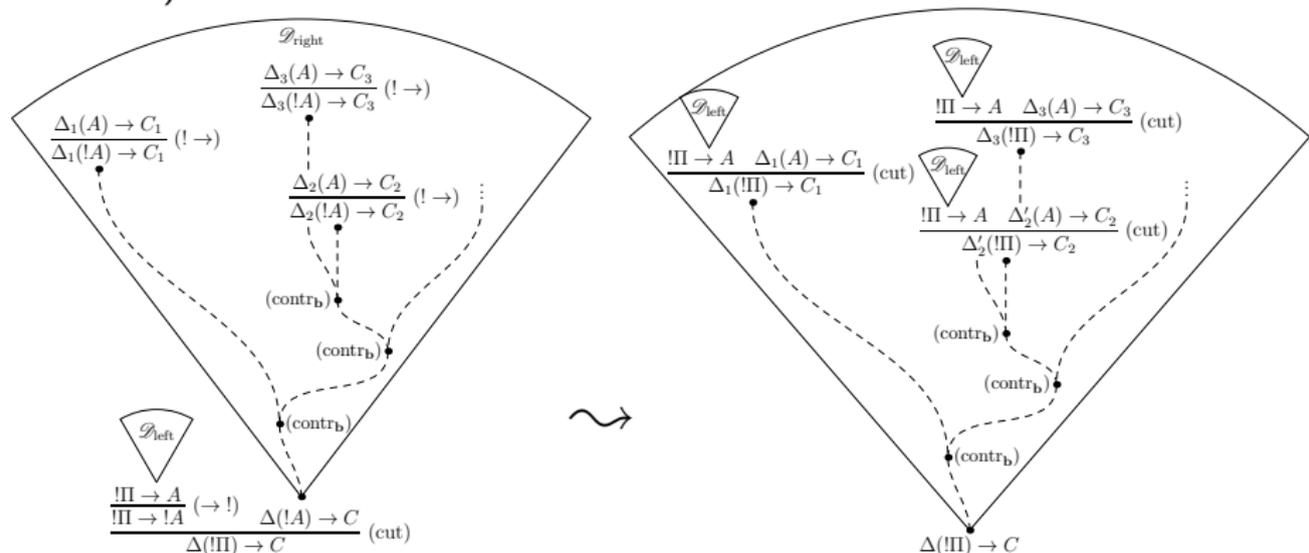
► Our analysis of syntactic phenomena is due to Morrill, 2011–2017.

Cut Elimination in $\mathbf{!}_b\mathbf{L}^1$

We use deep cut elimination strategy (cf. Braüner and de Paiva 1996).

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$$\frac{\Delta_1, ! \square^{-1} B, \Delta_2, B, \Delta_3 \rightarrow C}{\Delta_1, ! \square^{-1} B, \Delta_2, \Delta_3 \rightarrow C} \text{ (inst)}$$

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$$!\Gamma = !B_1, \dots, !B_n,$$

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$$!\Phi = !(1 / (!B_1)), \dots, !(1 / (!B_n)), \text{ and}$$

$$!\tilde{\Phi} = !(1 / (!\square^{-1}B_1)), \dots, !(1 / (!\square^{-1}B_n)).$$

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4. $s \Rightarrow^* a_1 \dots a_n$ in the type-0 grammar.

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Thank you !