Parameterized Aspects of Triangle Enumeration

Matthias Bentert Till Fluschnik André Nichterlein
Rolf Niedermeier

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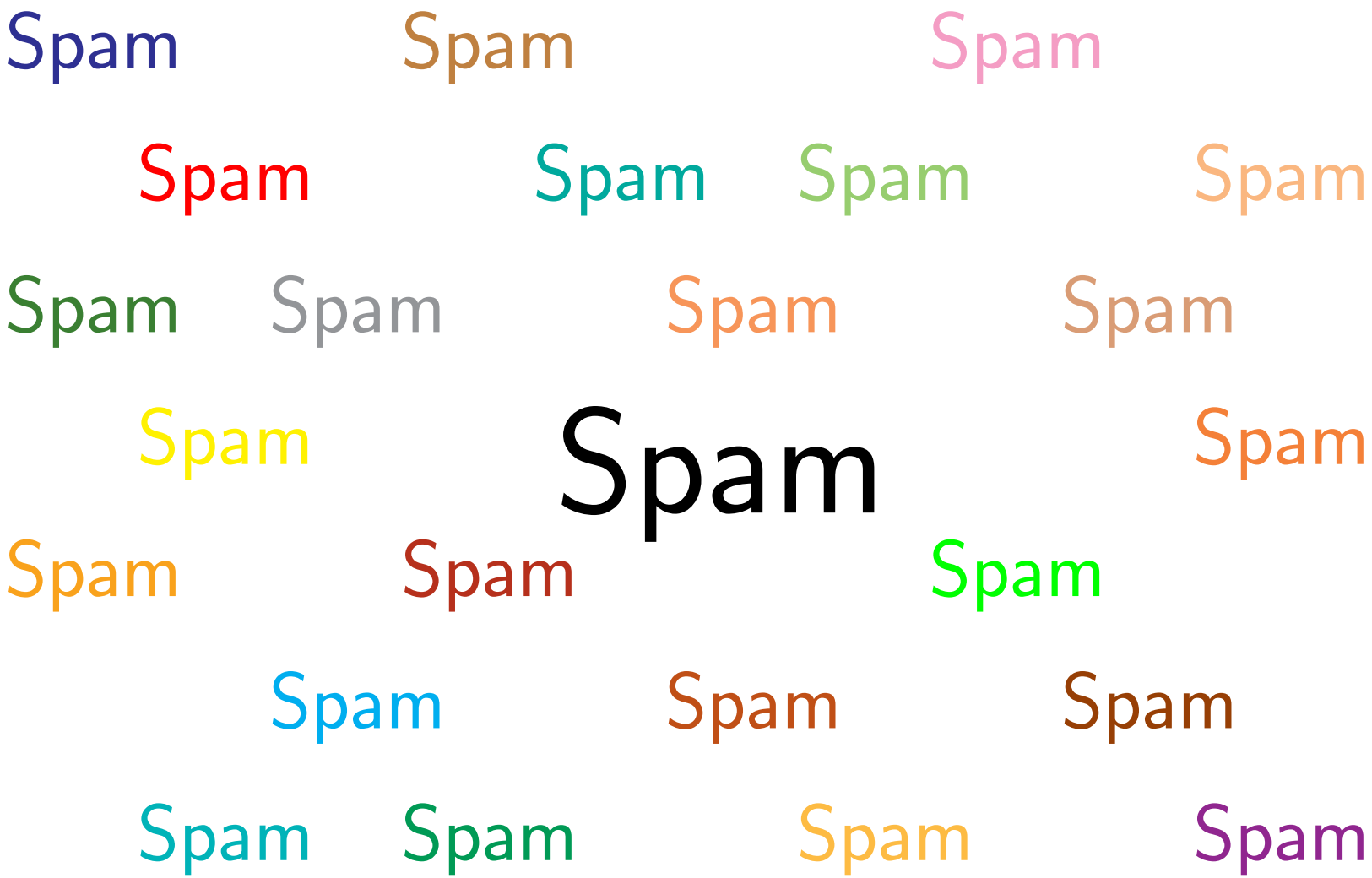
21st International Symposium on Fundamentals of Computation Theory, Bordeaux, September 12th, 2017

Spam Detection



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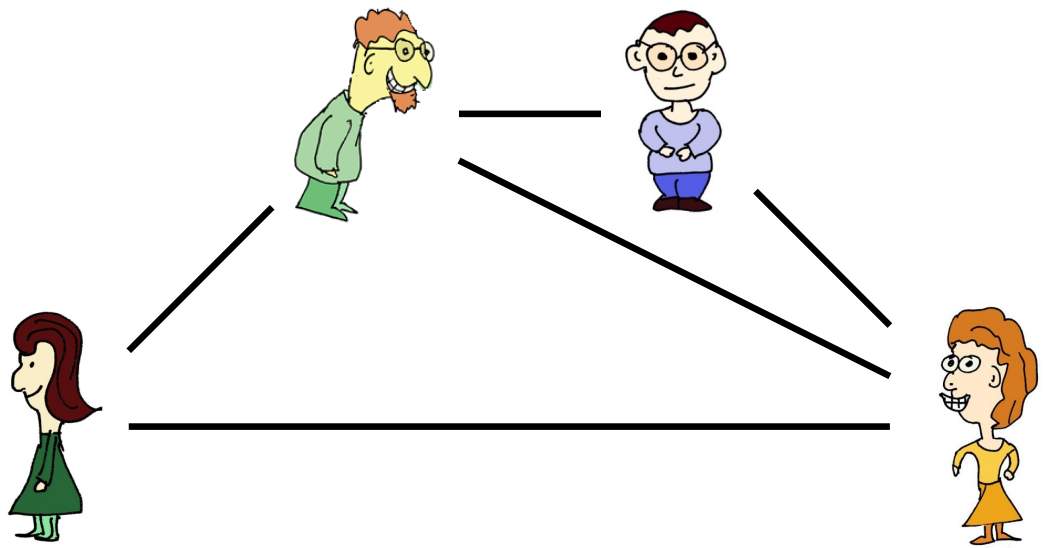

[Drawn by Piotr Faliszewski]

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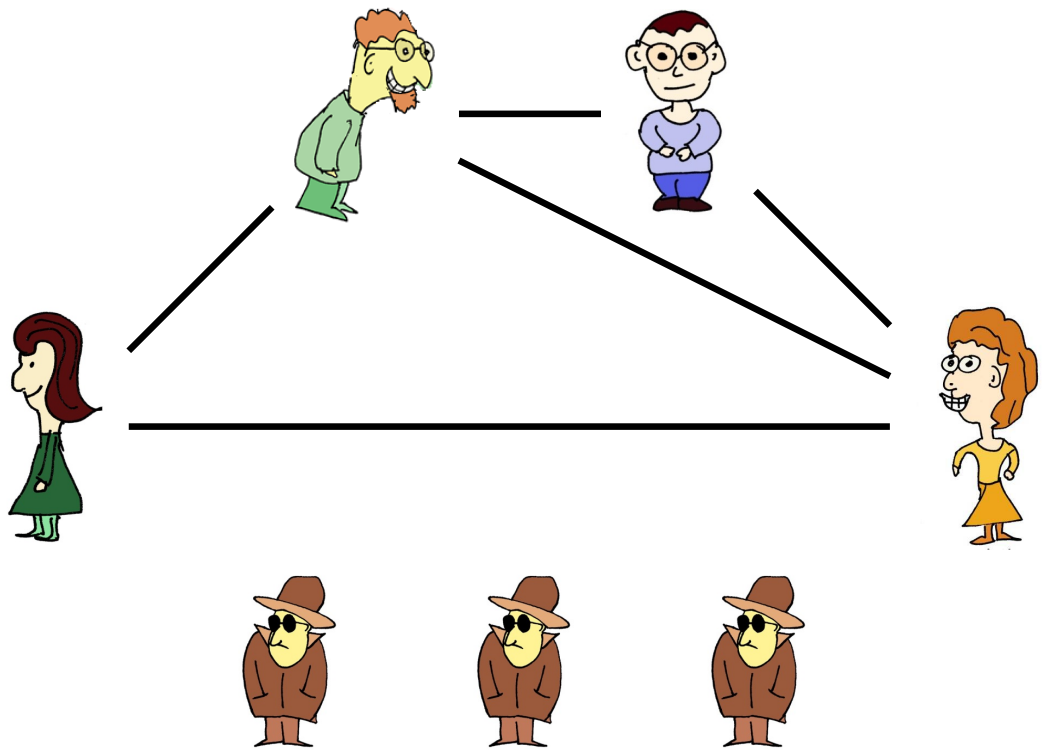

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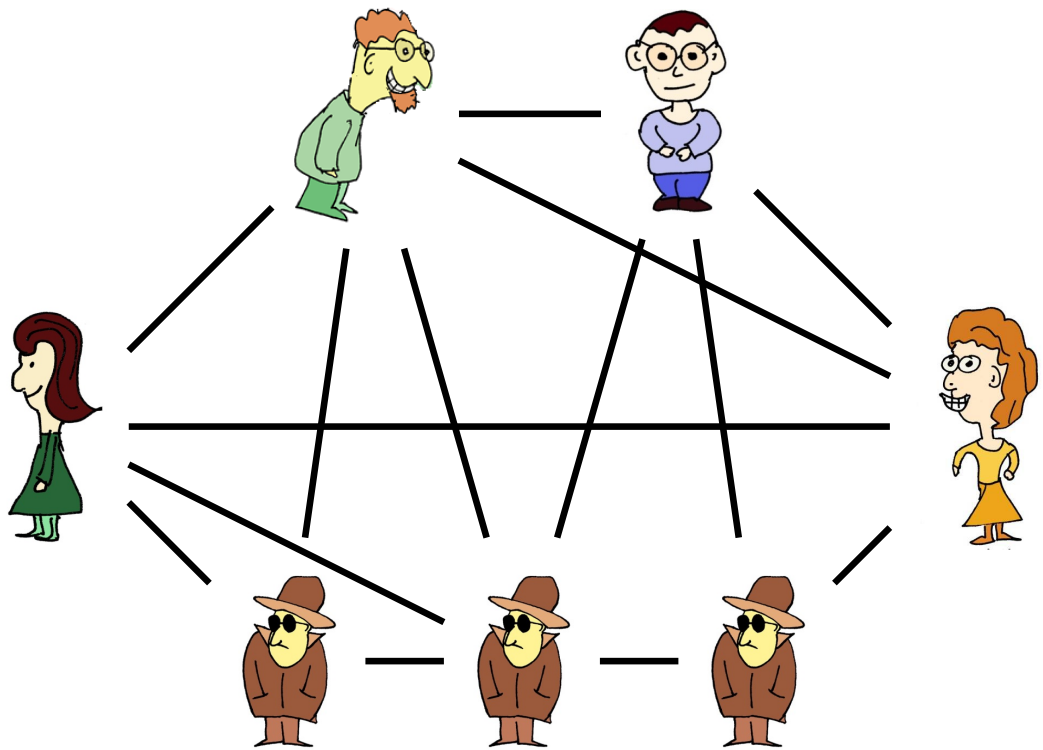

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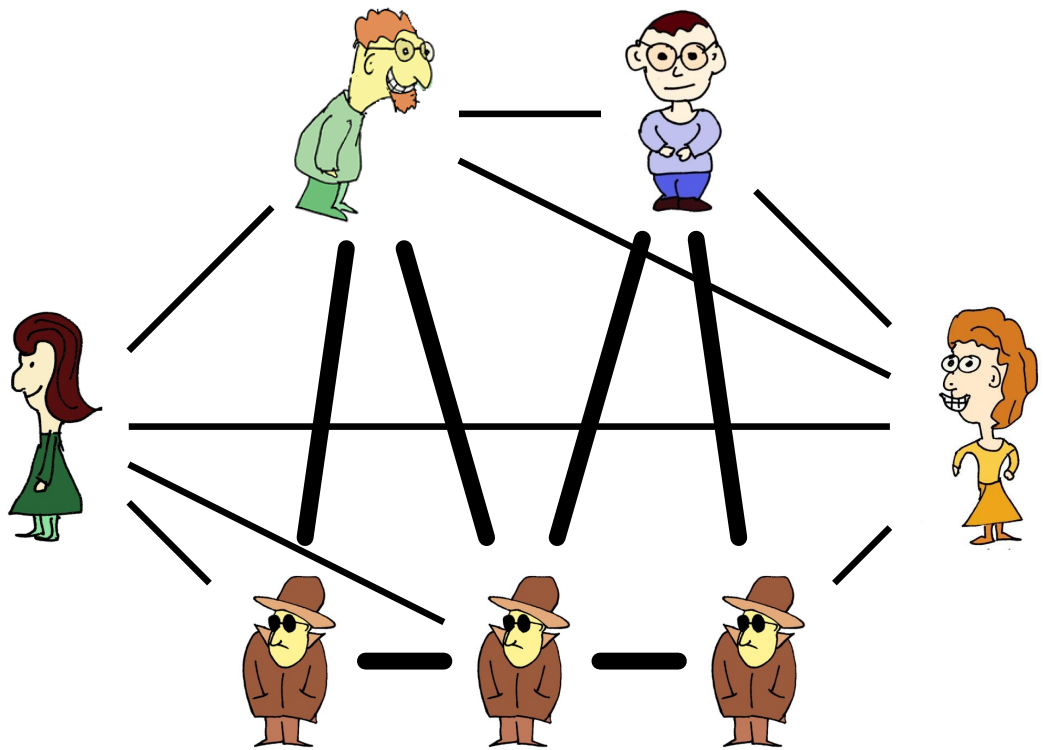

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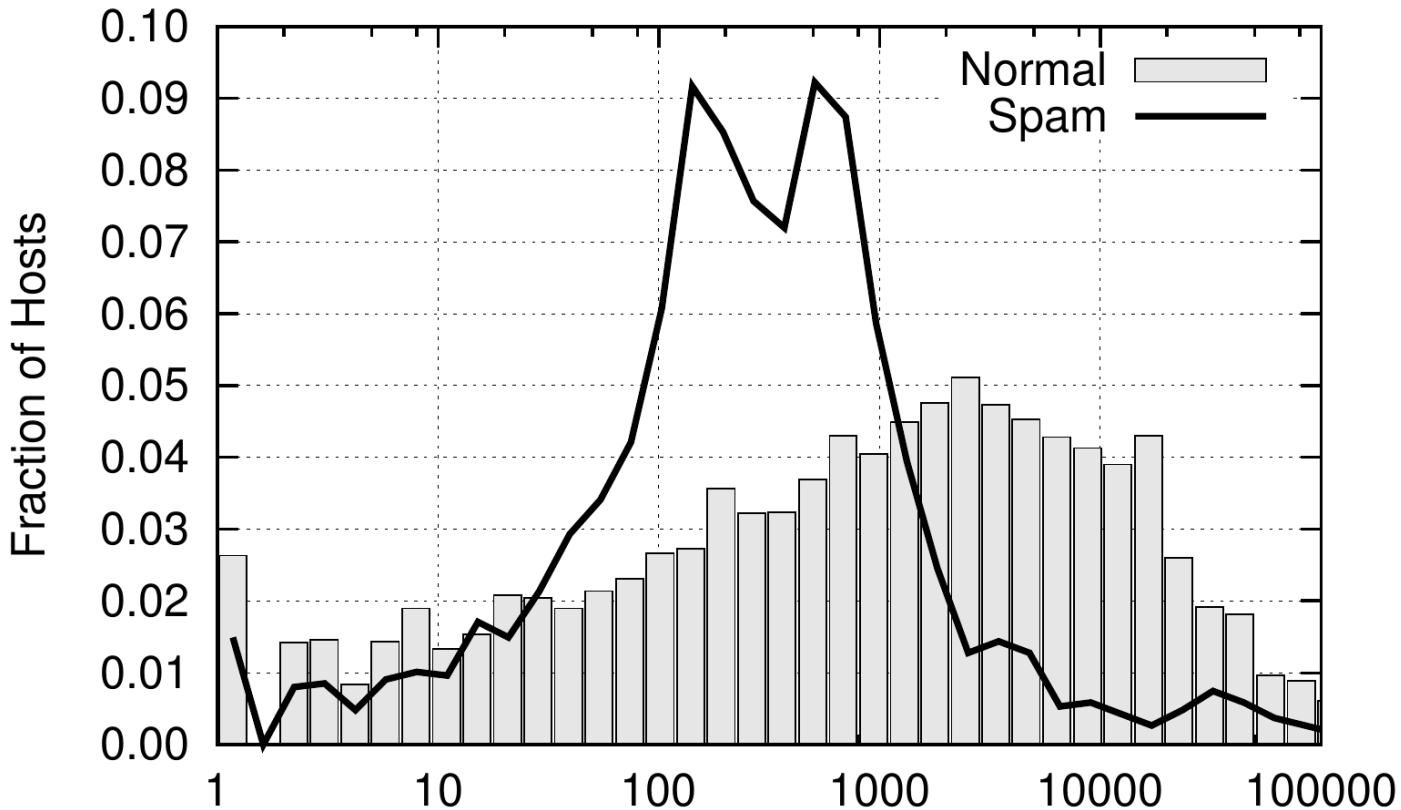
Spam Detection III



[Becchetti, L., Boldi, P., Castillo, C., Gionis, A.: Efficient Semi-Streaming Algorithms for Local Triangle Counting in Massive Graphs. ACM SIGKDD, 2008]

Further Applications

- Motif detection (computational biology)

[Yook, Oltvai, Barabasi: Proteomics, 2004]

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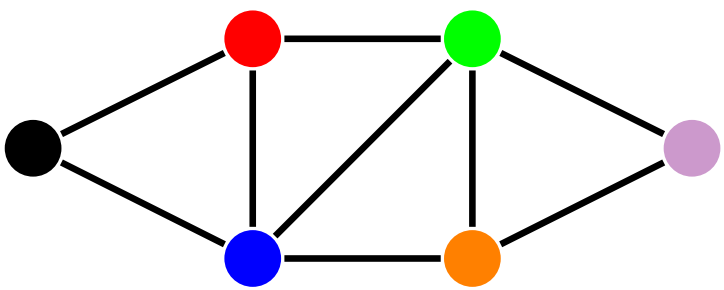

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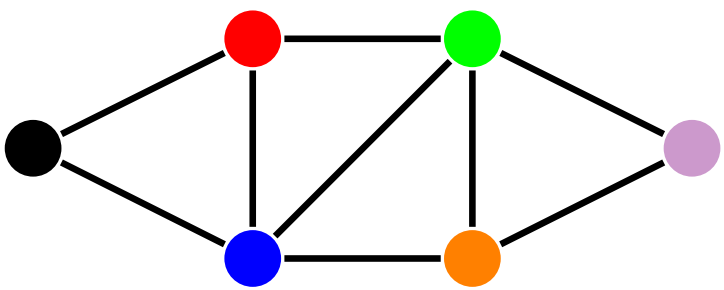

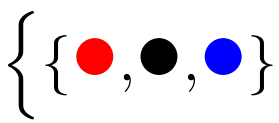

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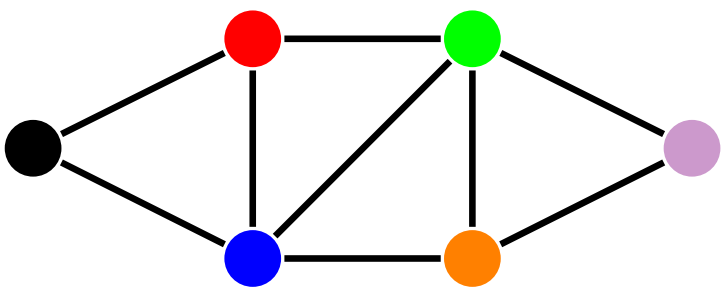

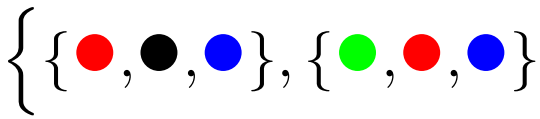

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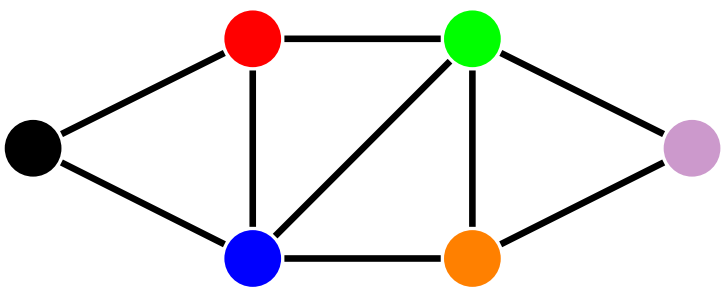

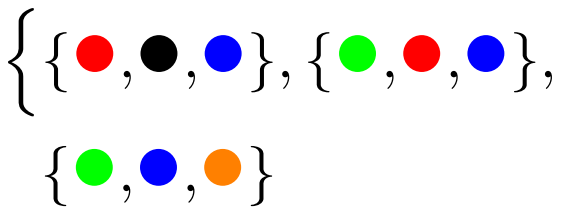

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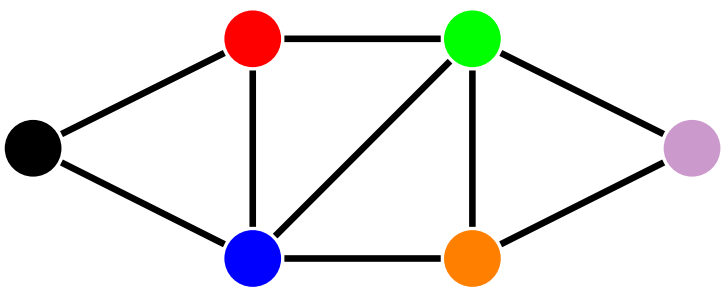

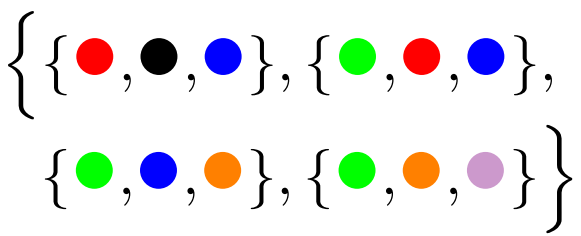

Related Work

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Our Results

Table: $\#\mathsf{T}$: number of triangles; Δ: maximum degree

parameter k	result

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	parameter k	result
solving		
kernel		
hard		

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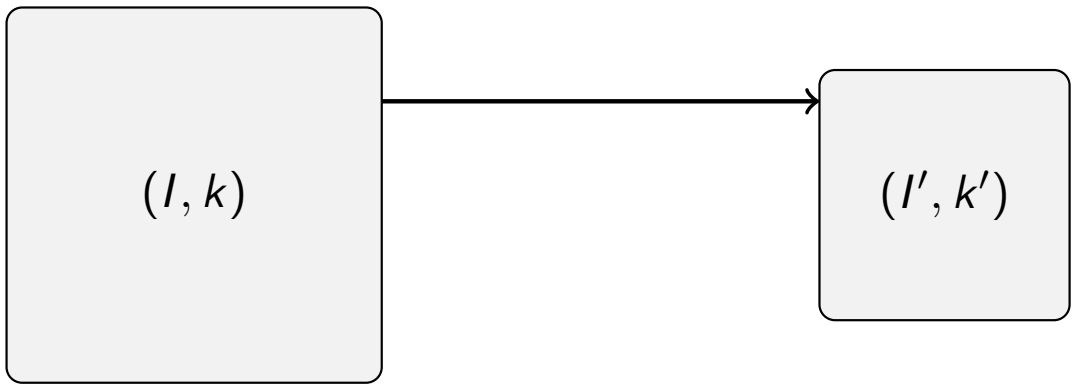
Kernelization

Kernels (for Decision Problems)



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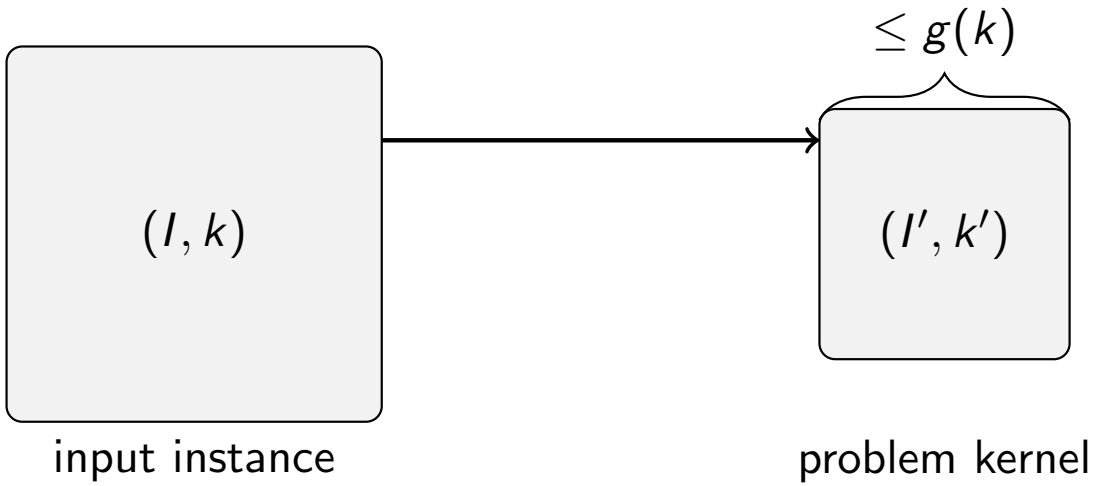

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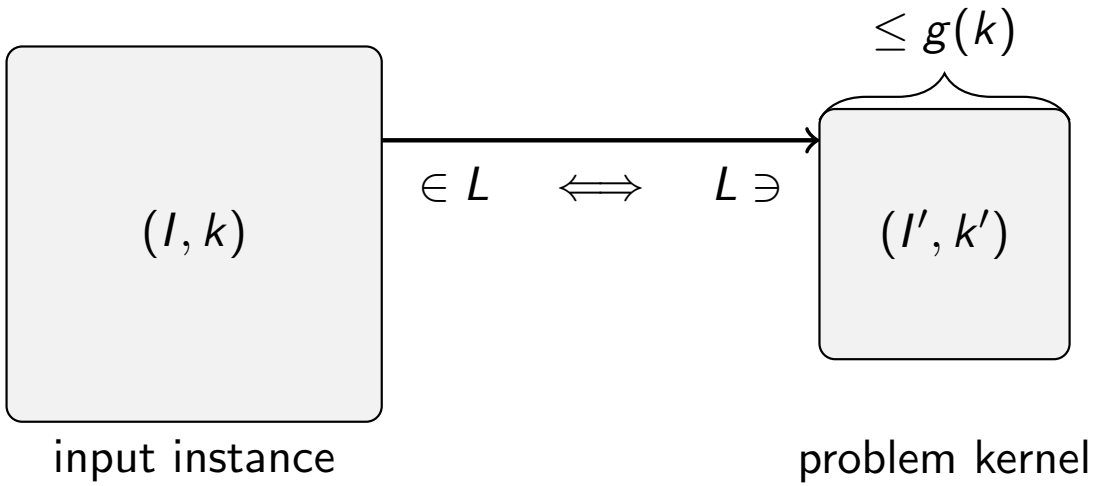

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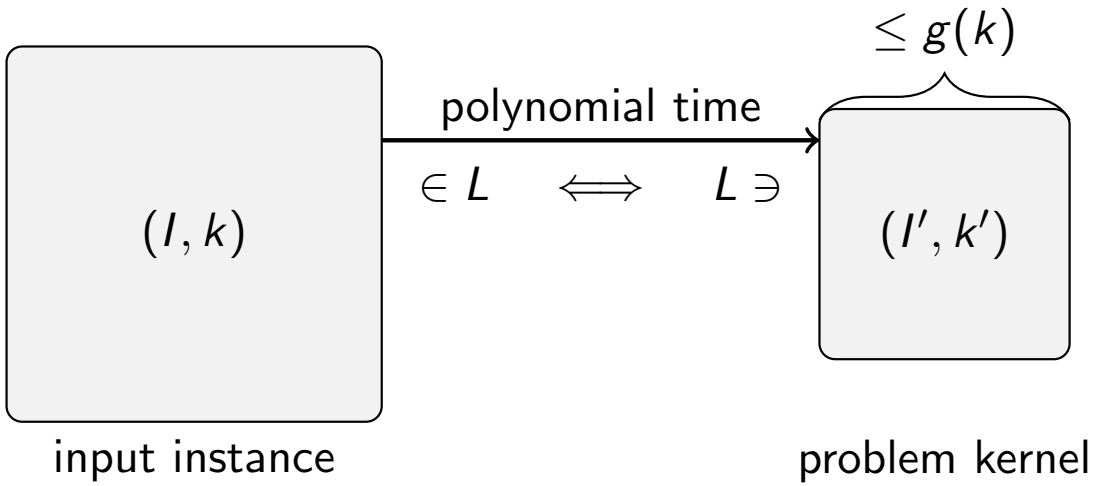

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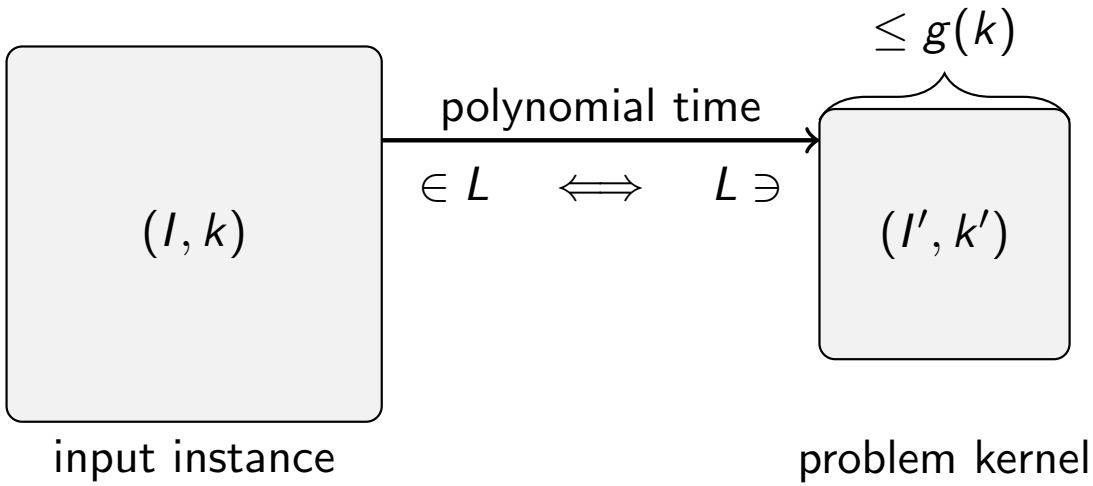

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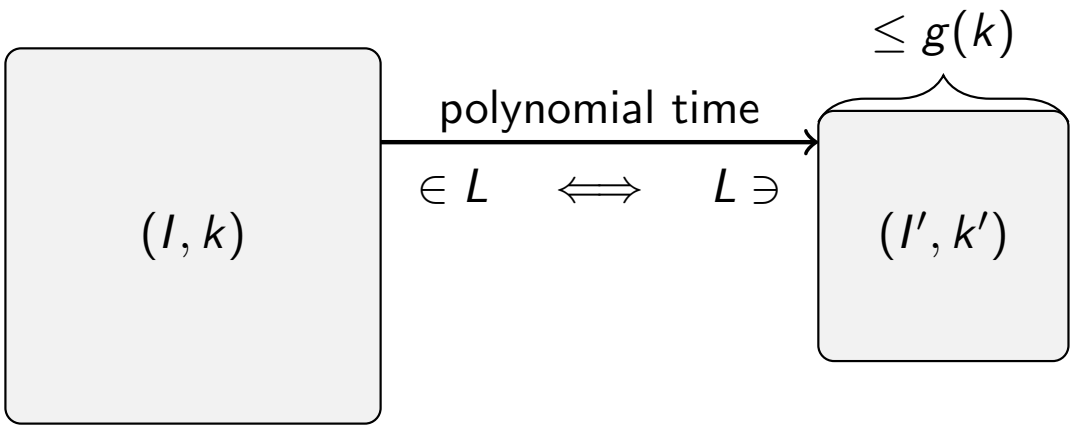
Enum-Kernels (for Enumeration Tasks)



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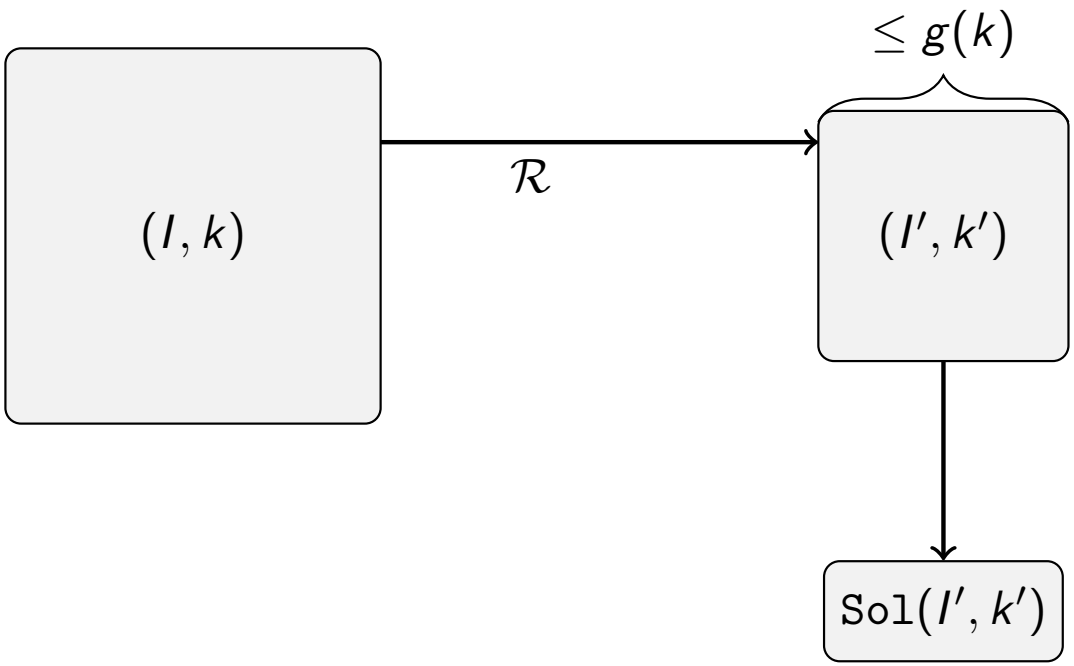

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$\texttt{Sol}$	(all) solutions

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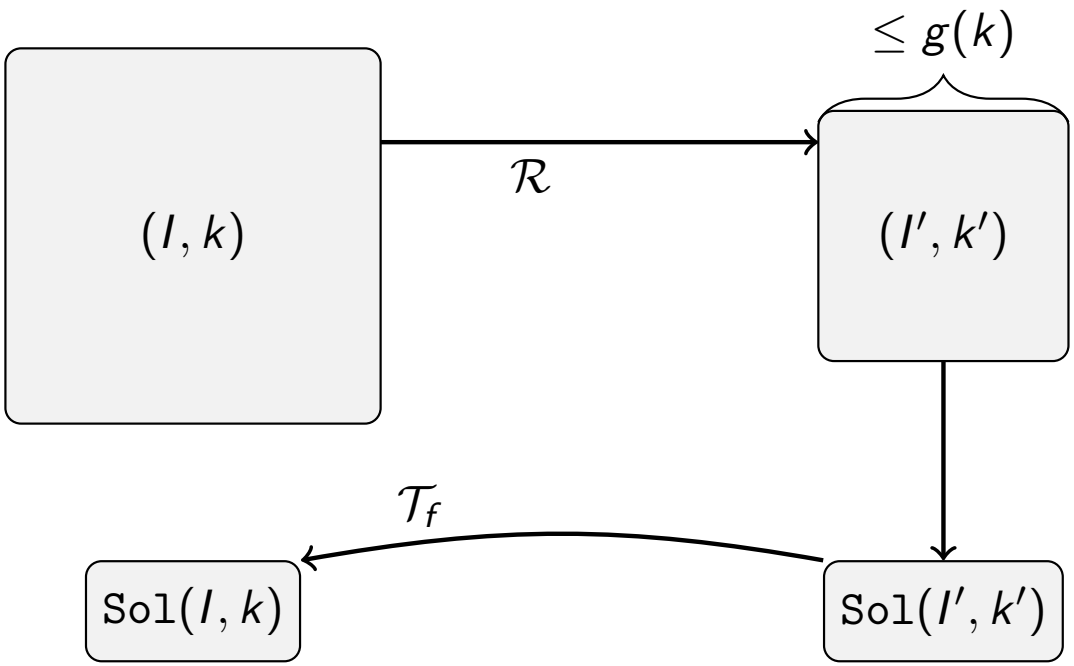

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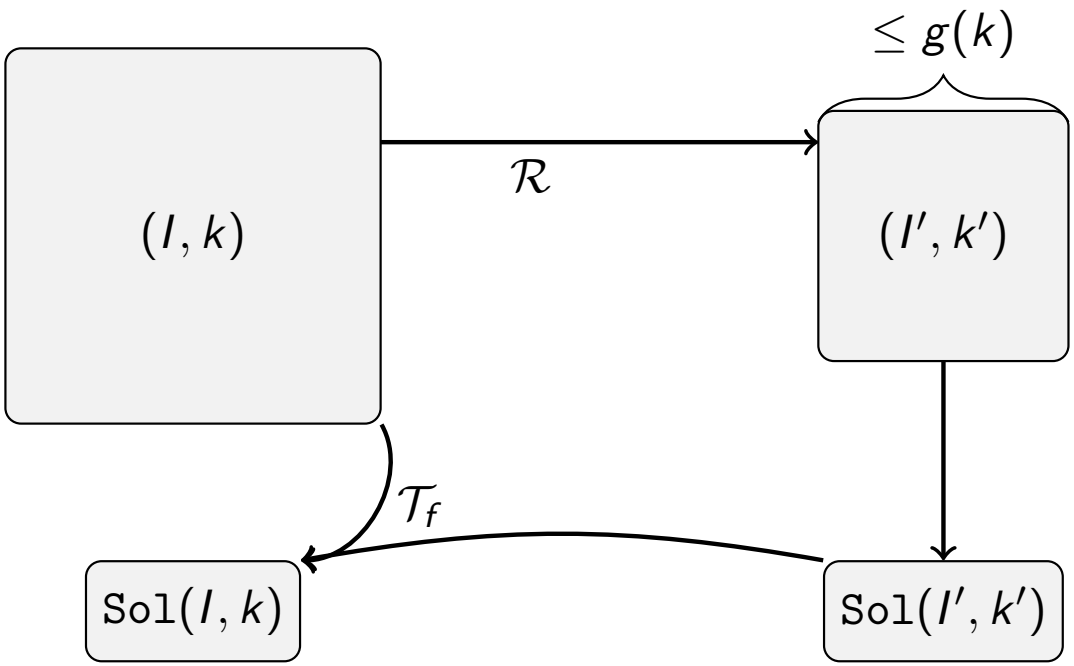

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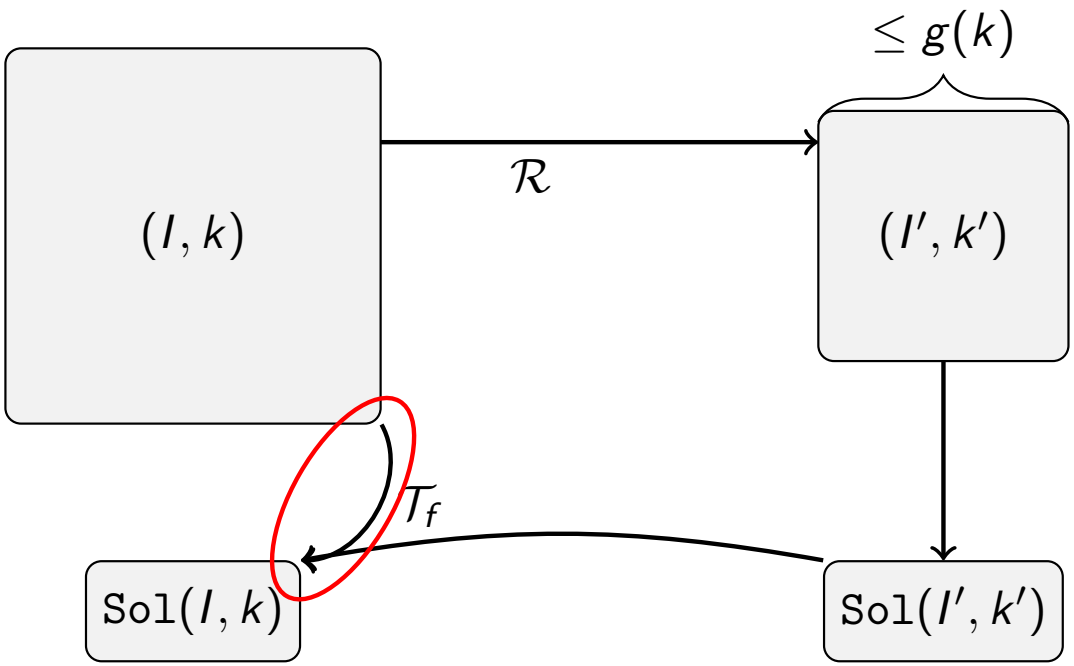

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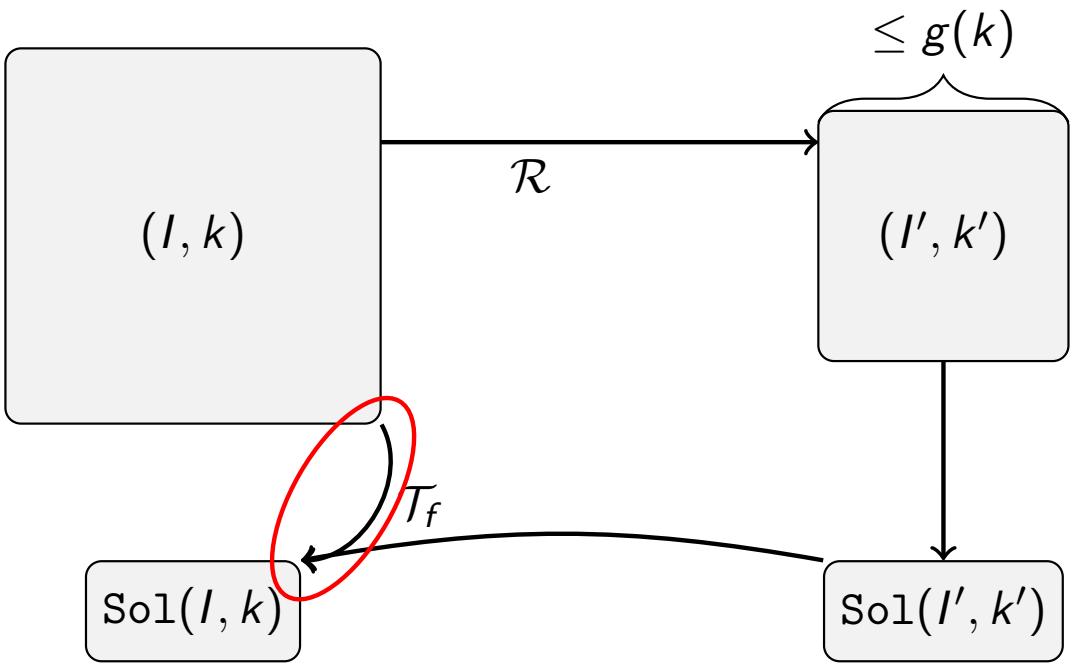

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Disadvantage: Kernel instance is insufficient!
(input instance is still needed)

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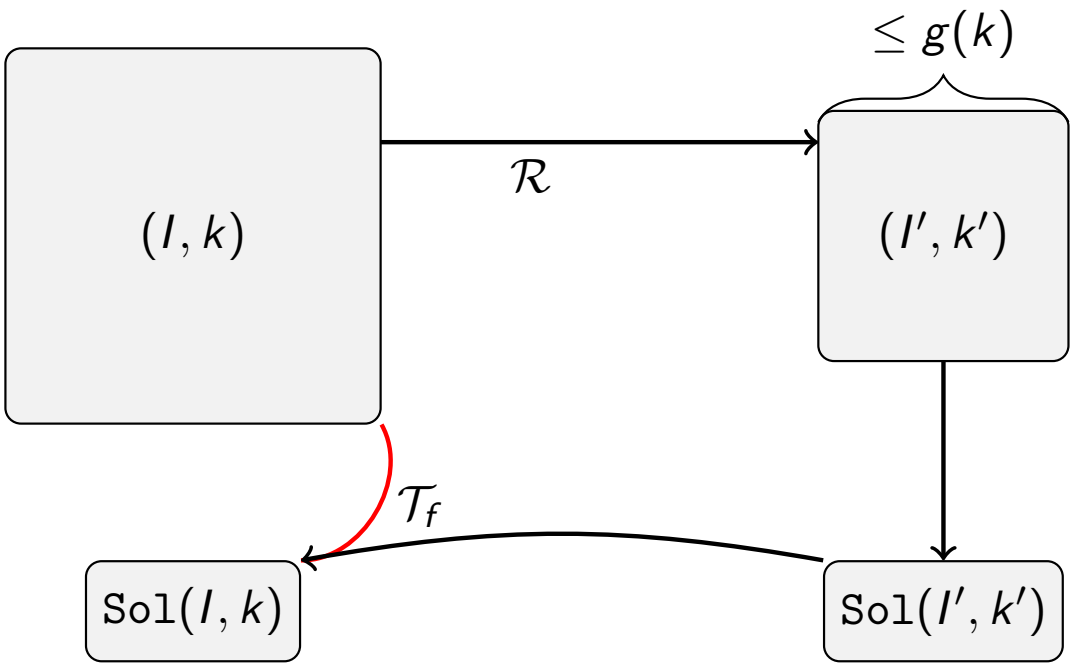

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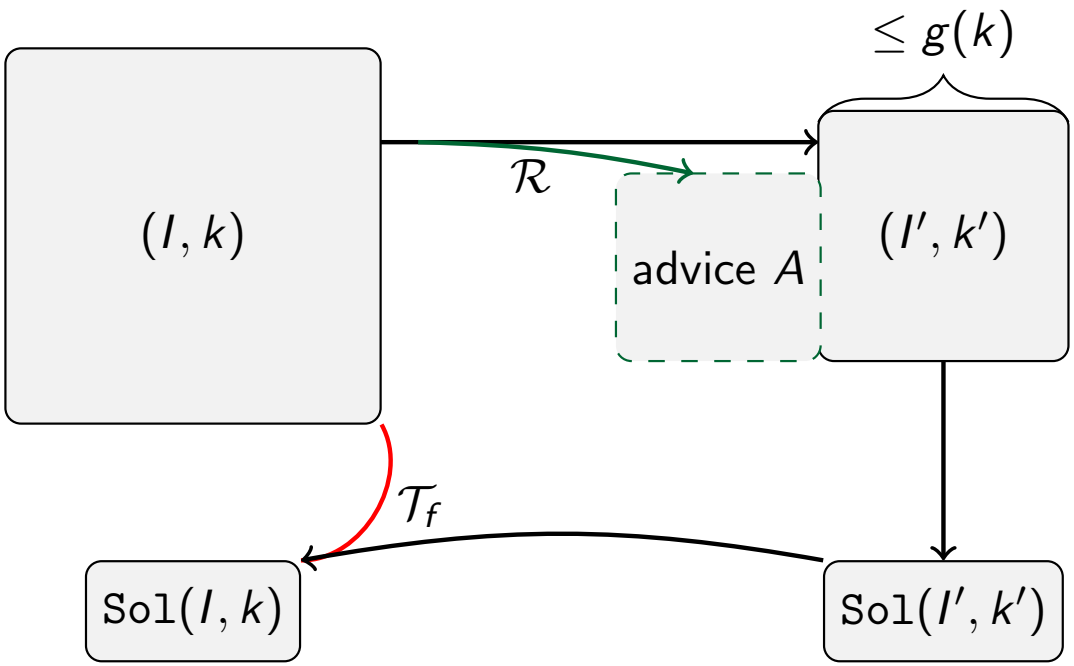

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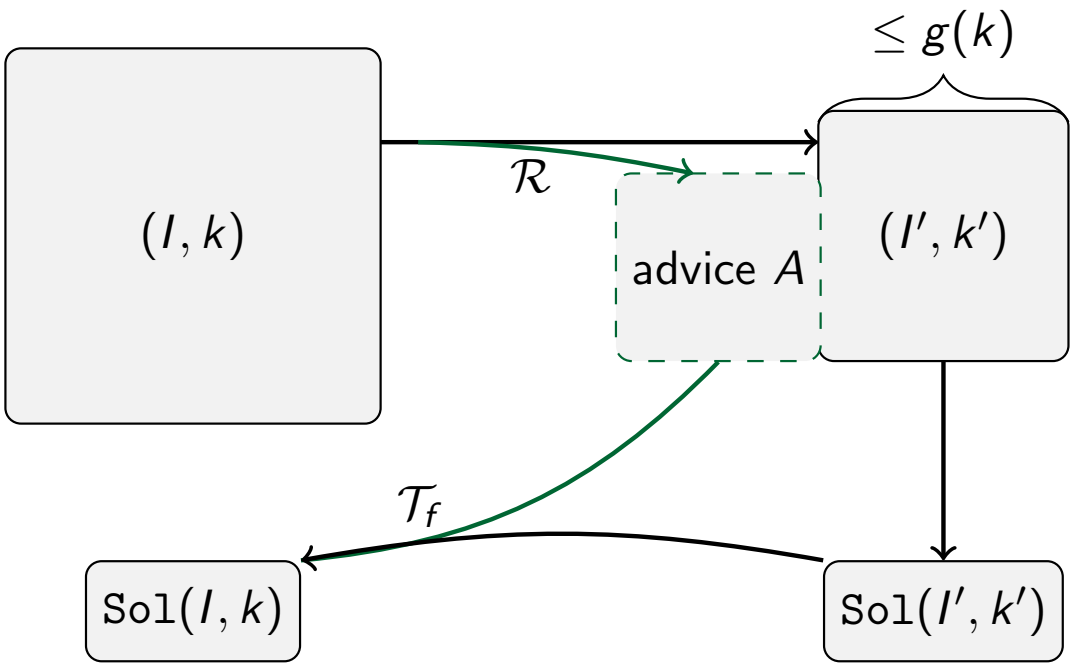

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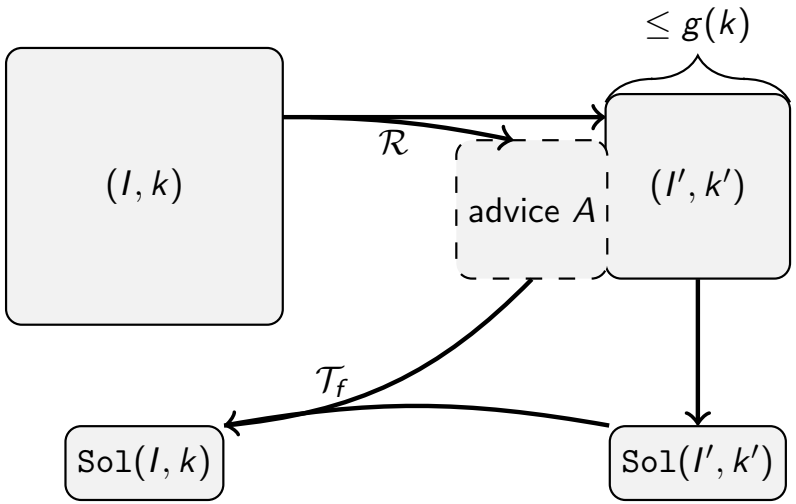

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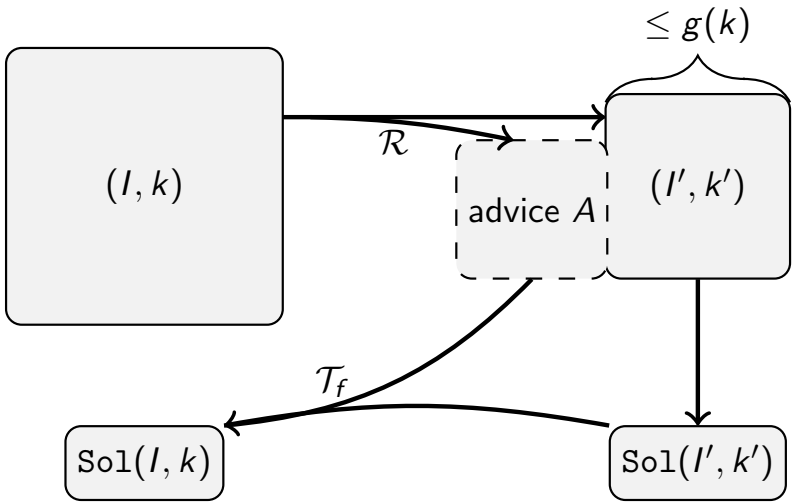

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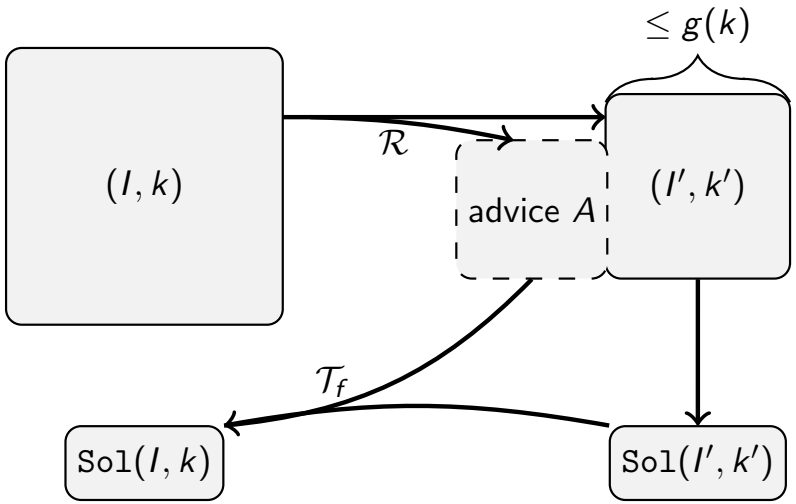

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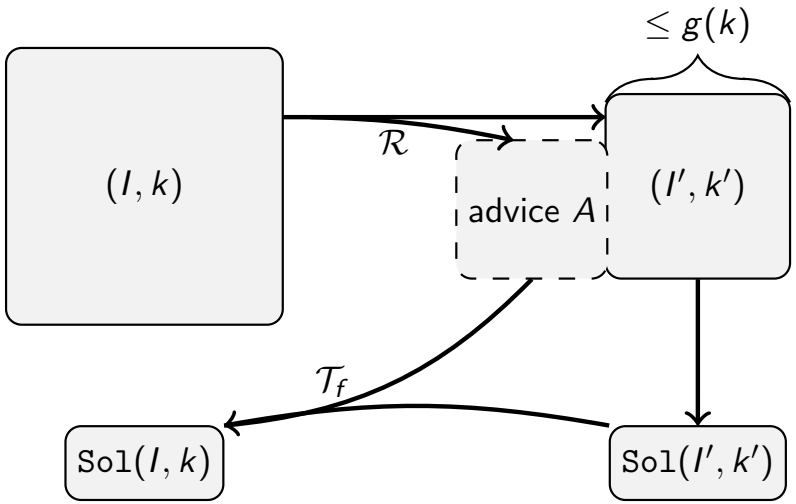

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- $\mathcal{R}$ and $\mathcal{T}_f$ can be computed fast enough.

Distance to d-Degenerate Graphs

Definition (degeneracy)

A graph has degeneracy d if every subgraph has a vertex of degree at most d.

Distance to d-Degenerate Graphs

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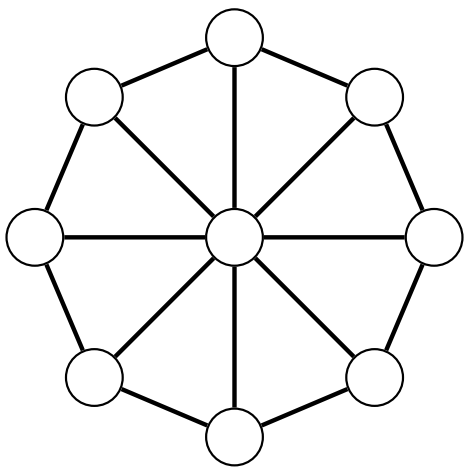

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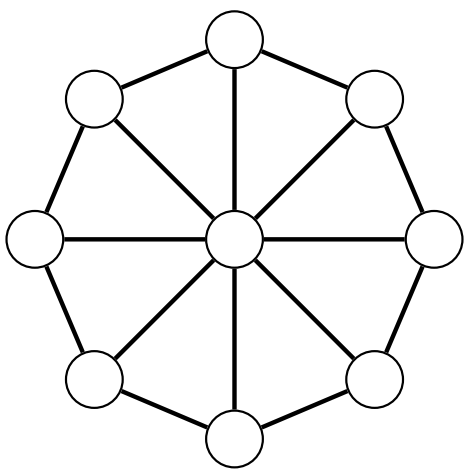

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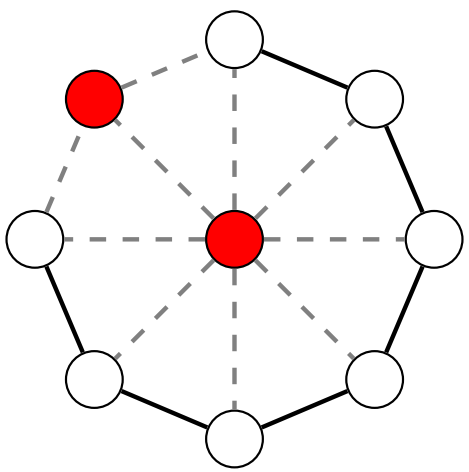

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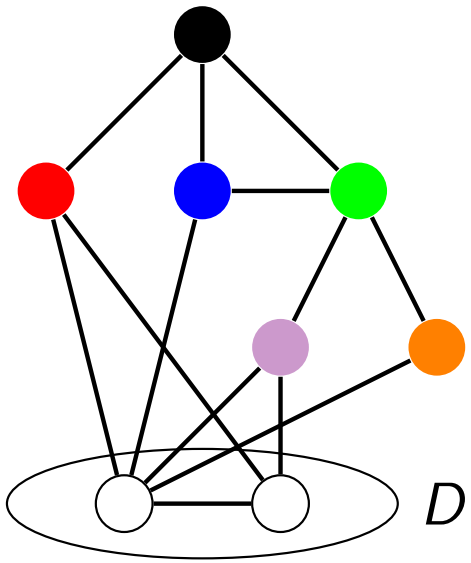
Enum-Advice Kernel for almost d-Degenerate Graphs



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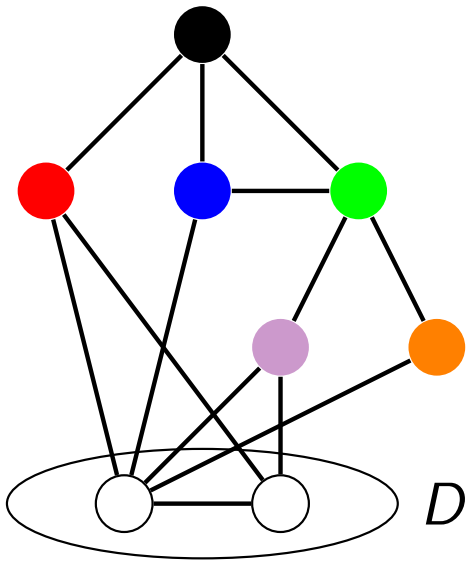

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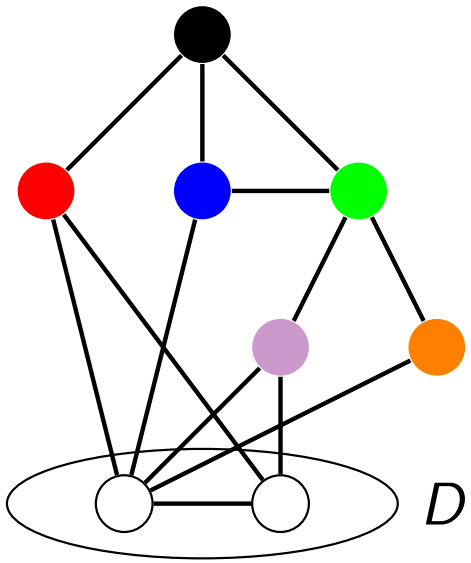

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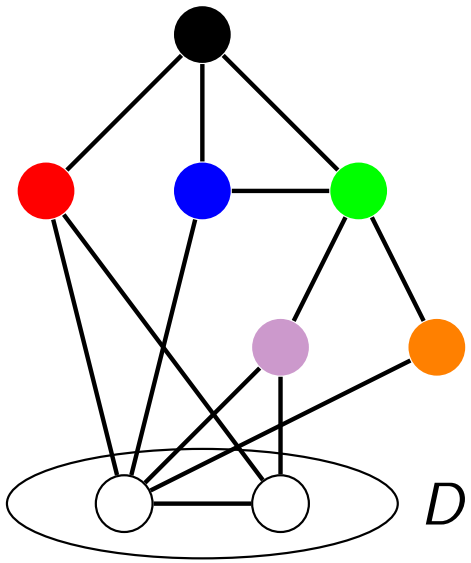

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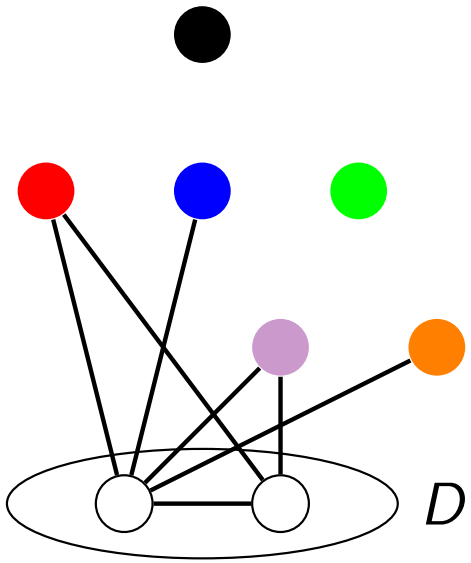

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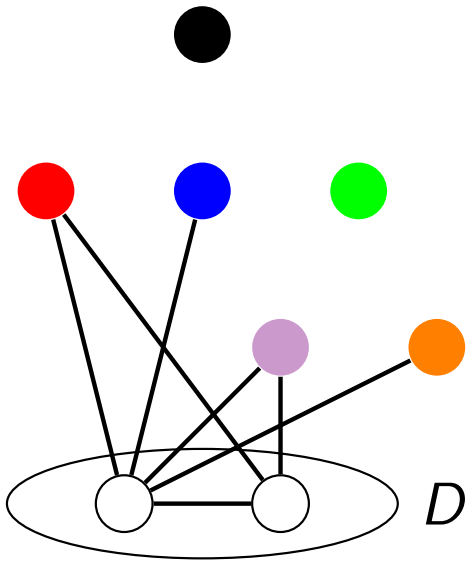

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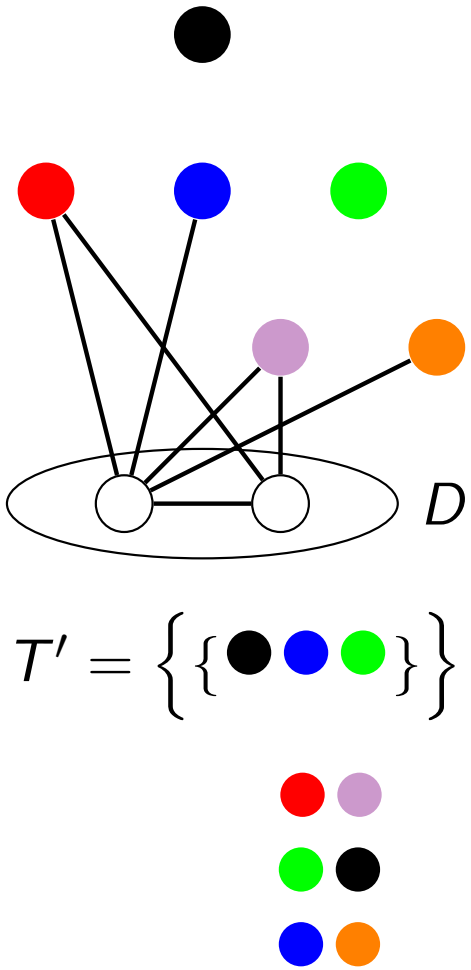

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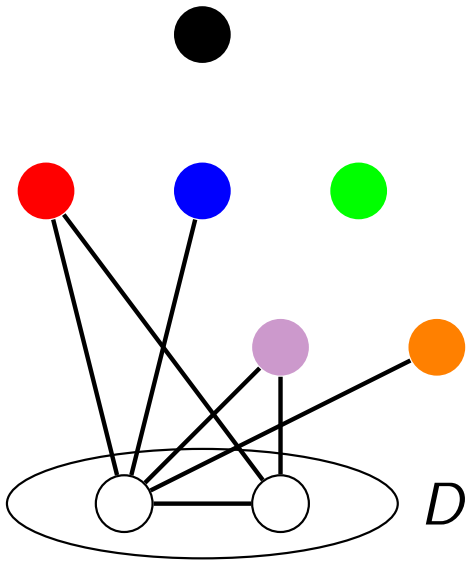
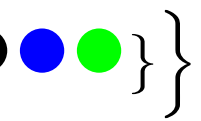
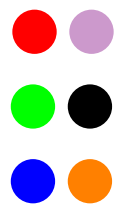


$T' =$ 



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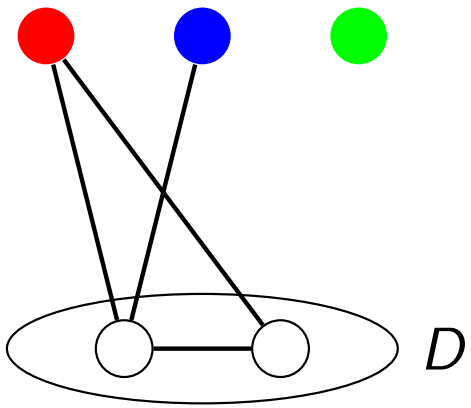

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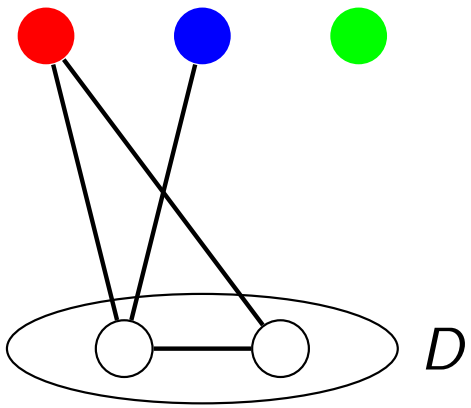
- Set parameter $k' = k$ and advice $A = (T', M)$.



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Computational Hardness

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($\rightsquigarrow$ Triangle Detection)

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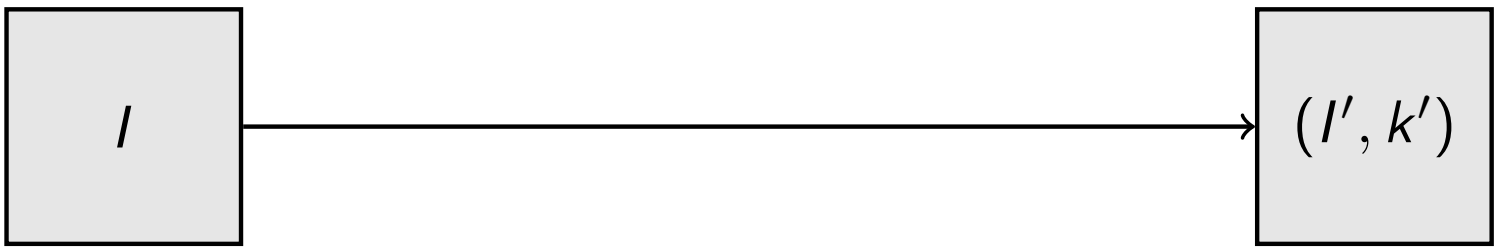
We call a problem L c-**General-Problem-hard**(f) if there exists an algorithm $\mathcal{A}$ transforming any input instance I of L into a new instance (I', k') of a parameterized version of L such that

General-Problem-Hardness



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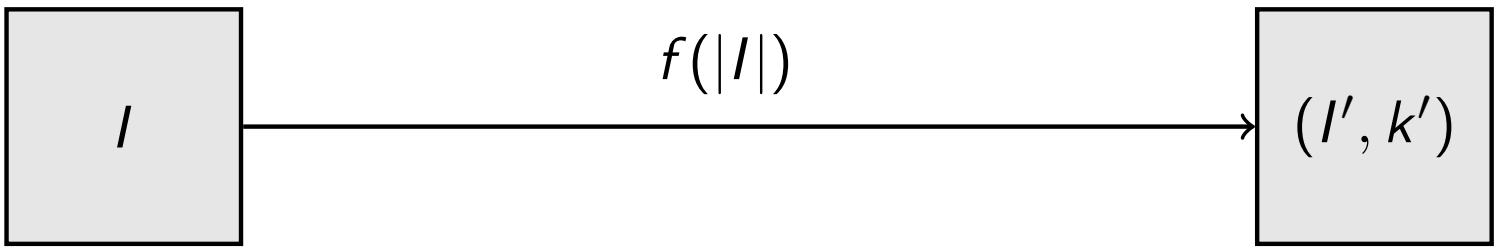

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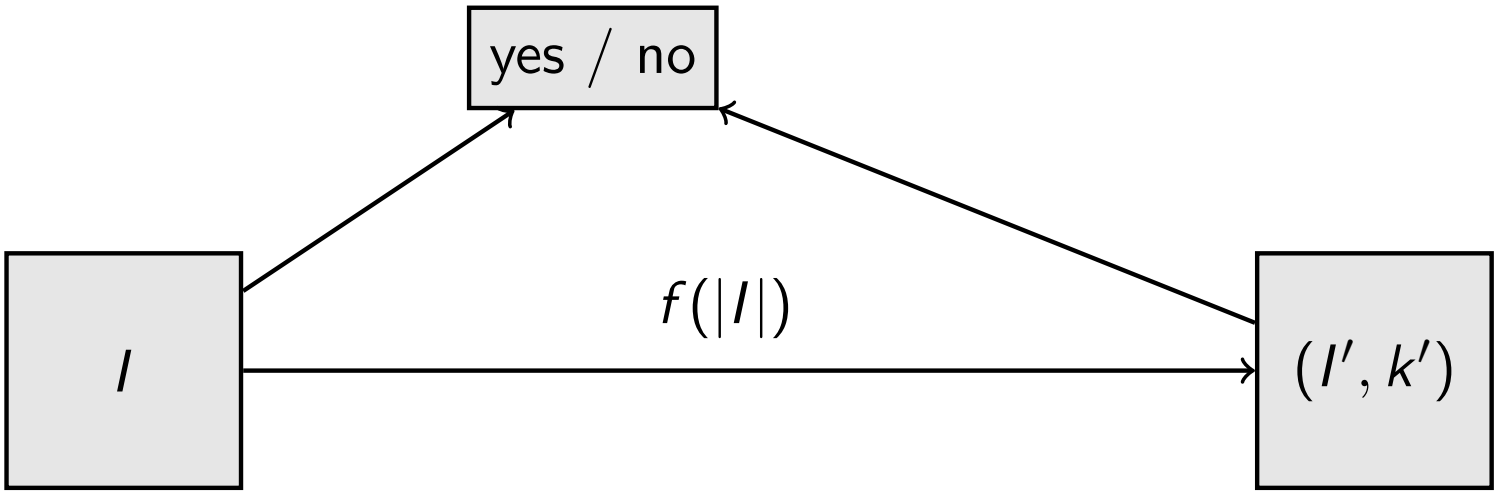

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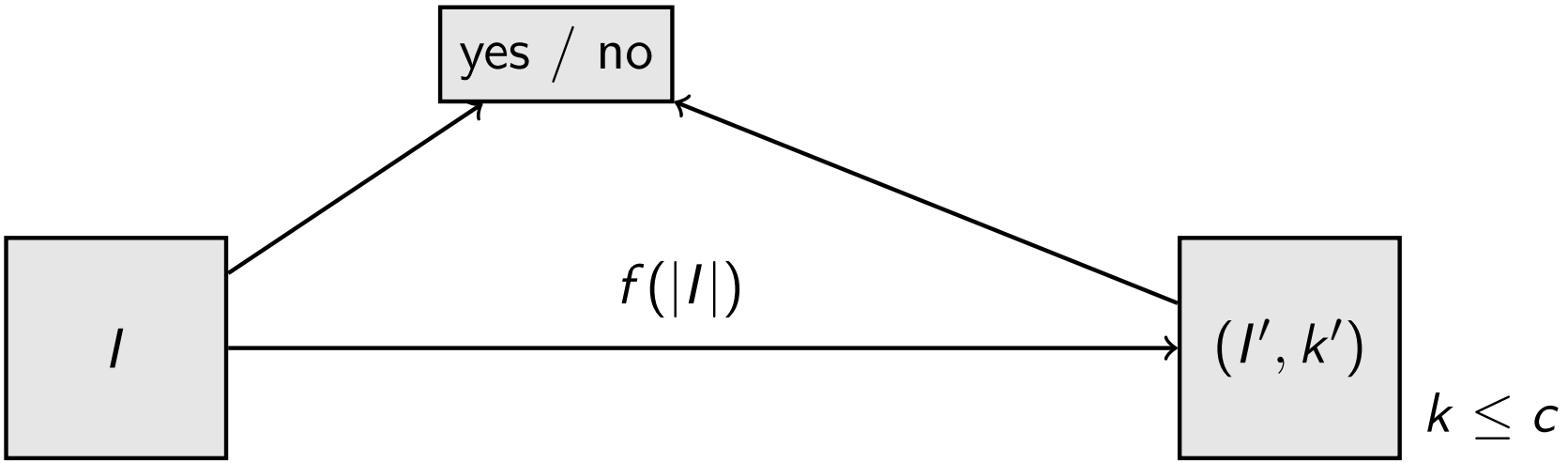

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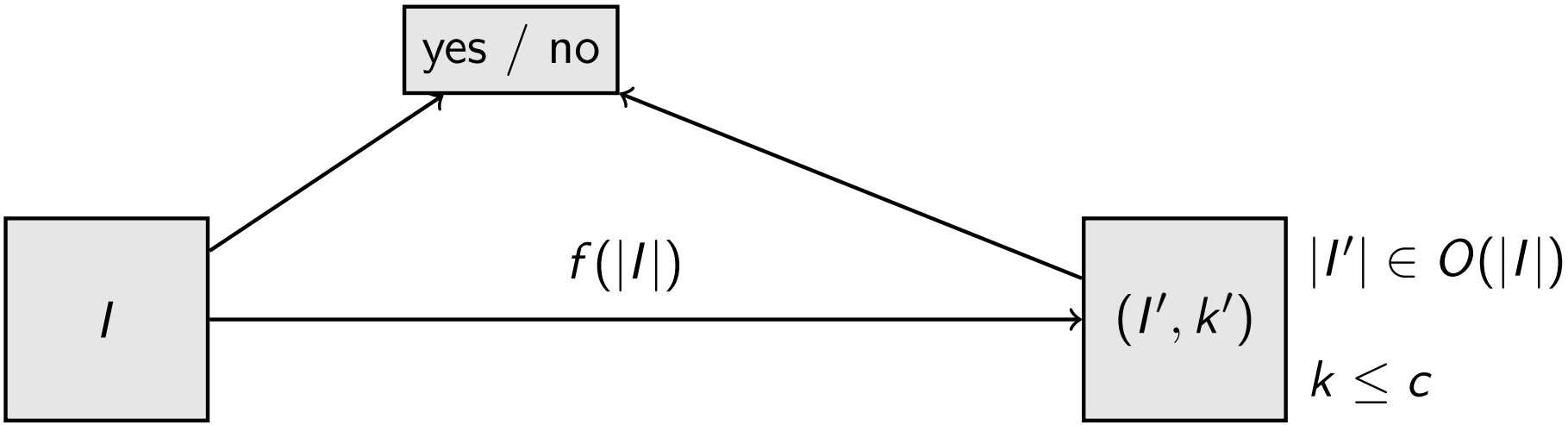

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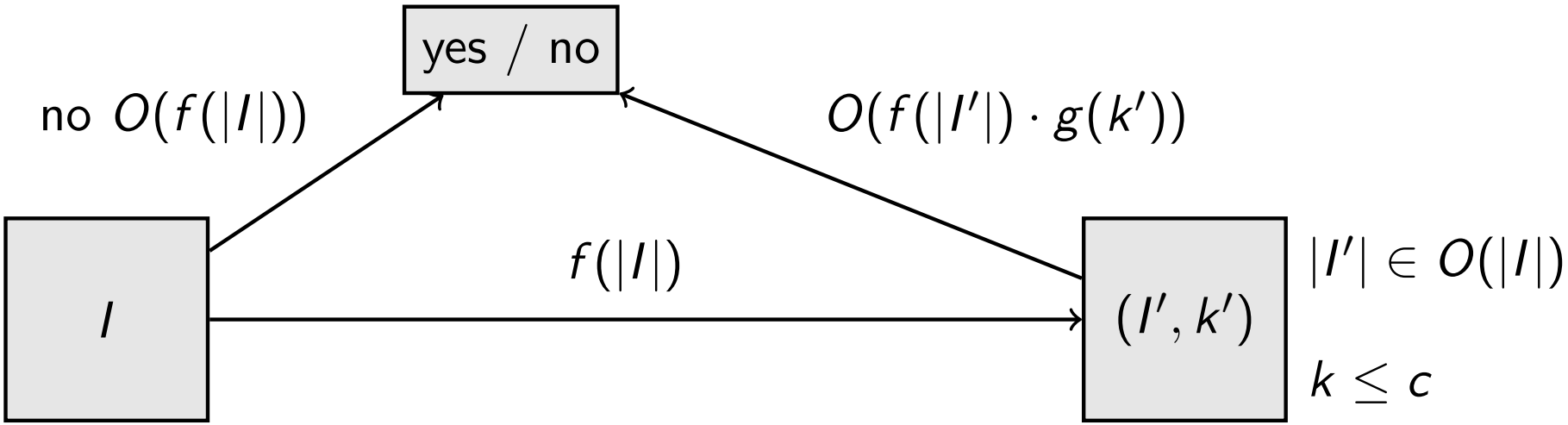

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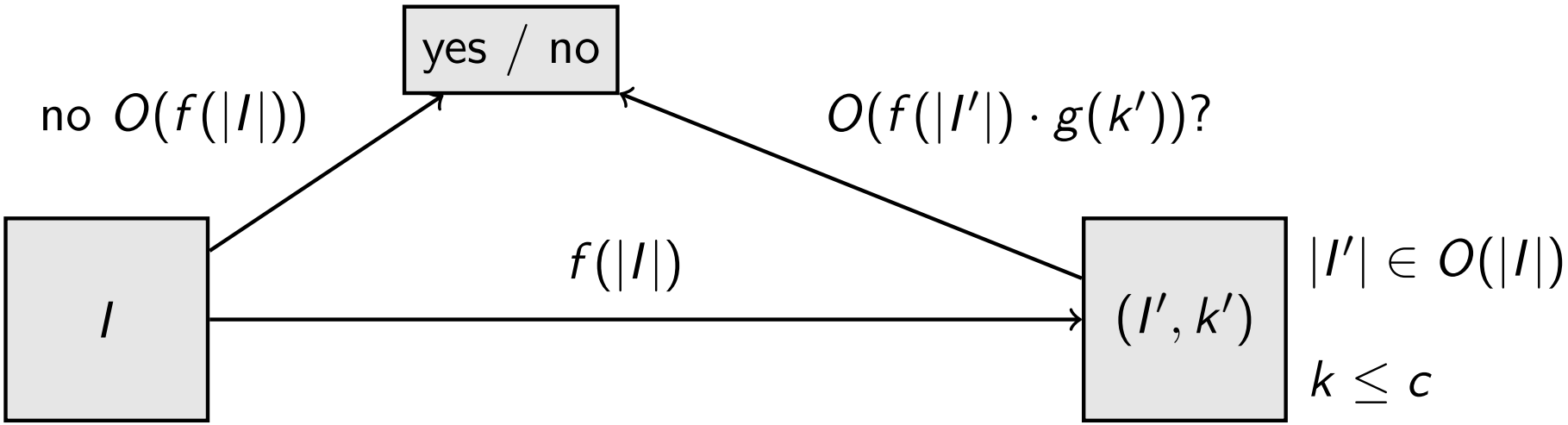

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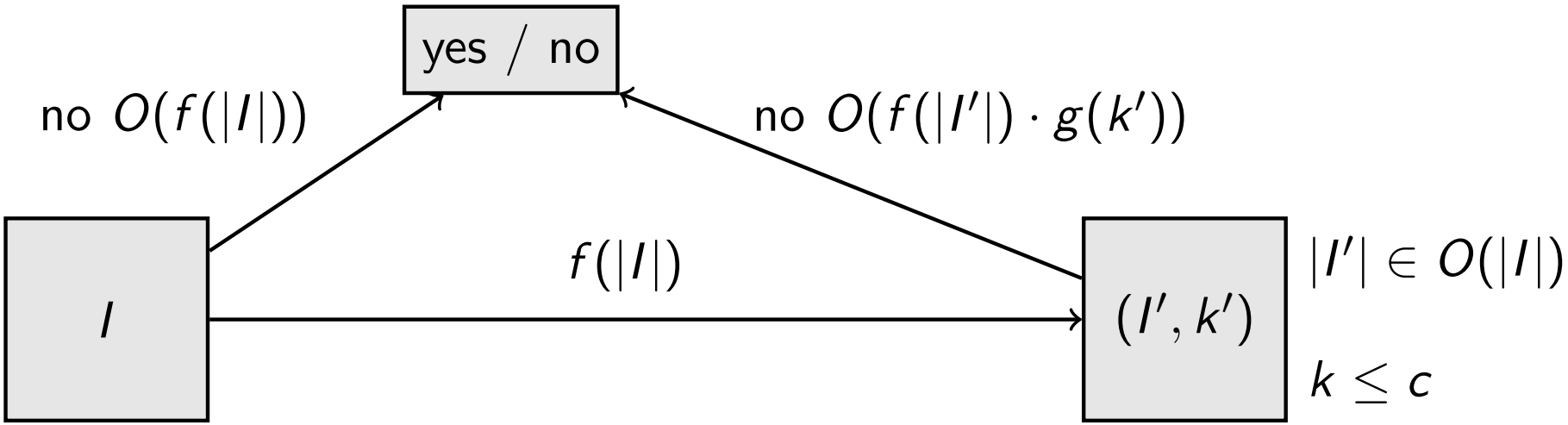

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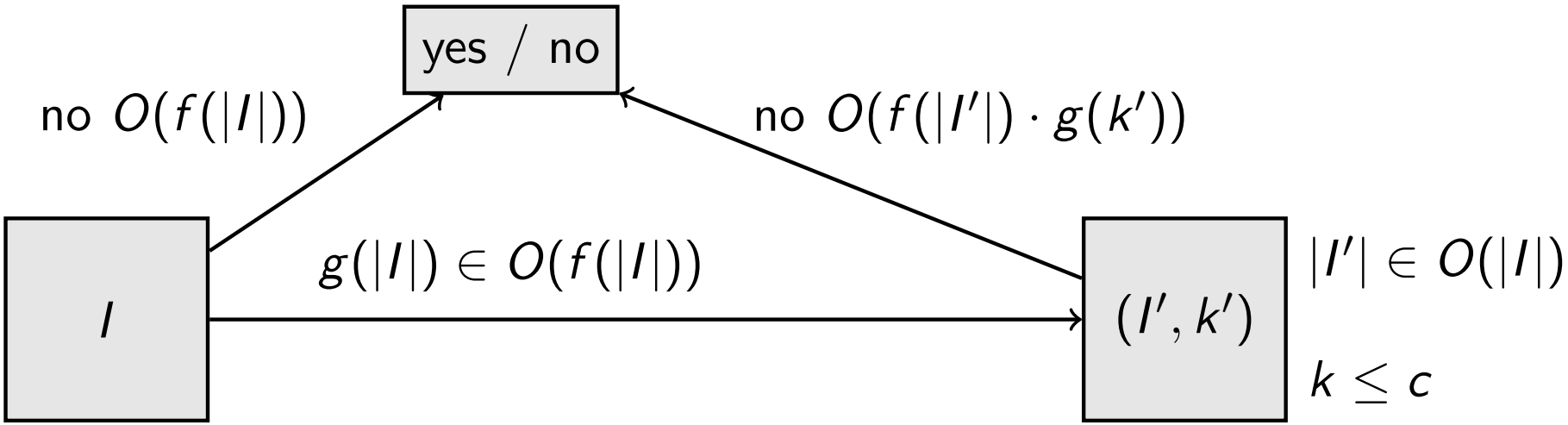

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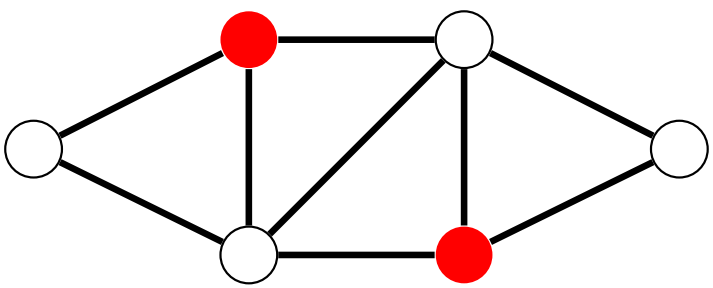

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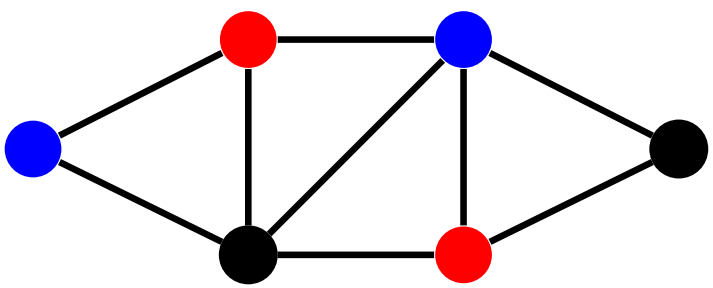

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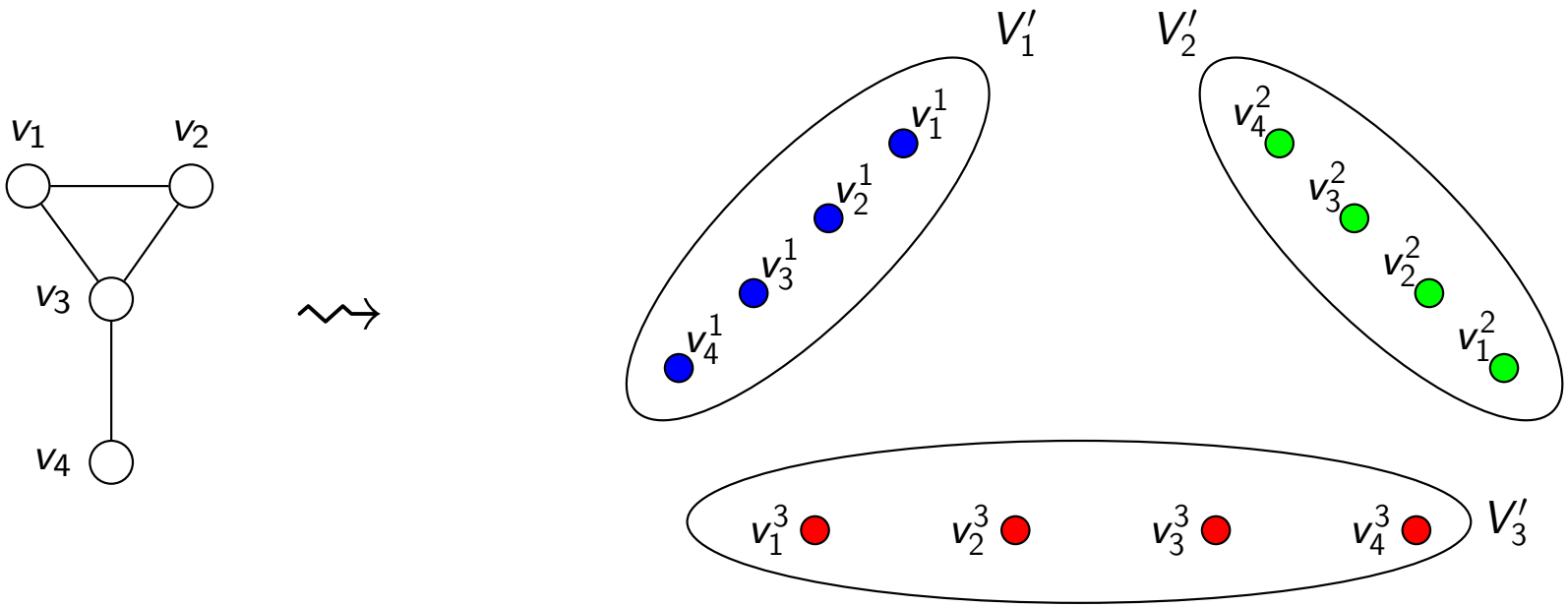

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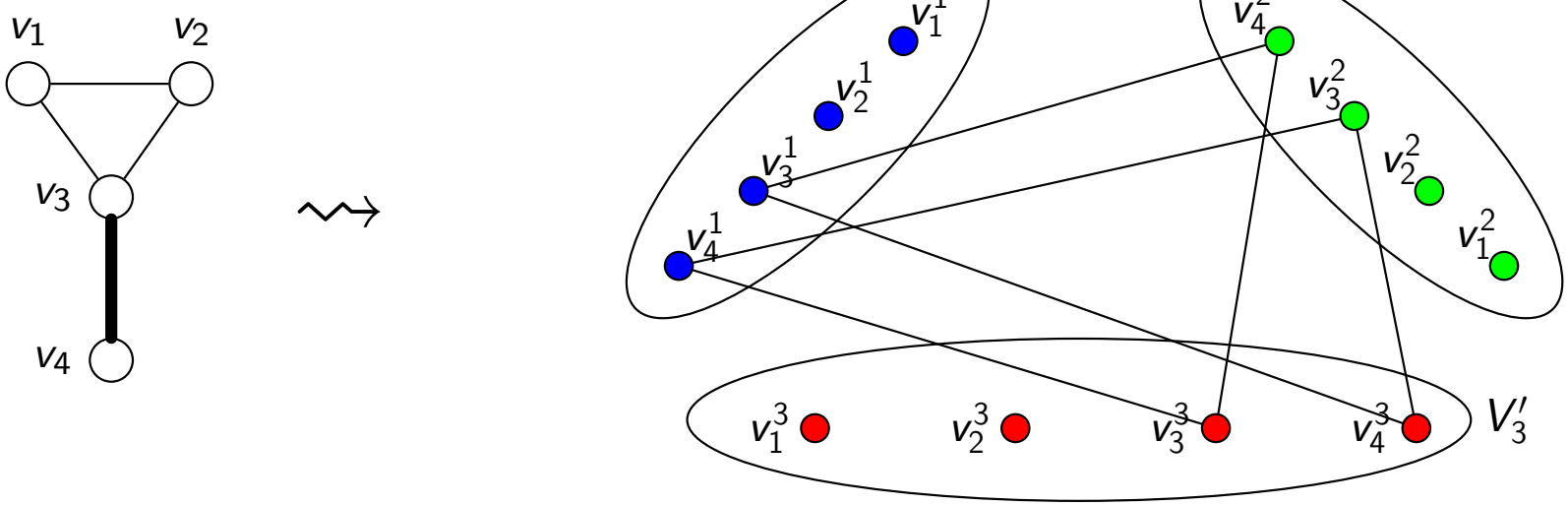

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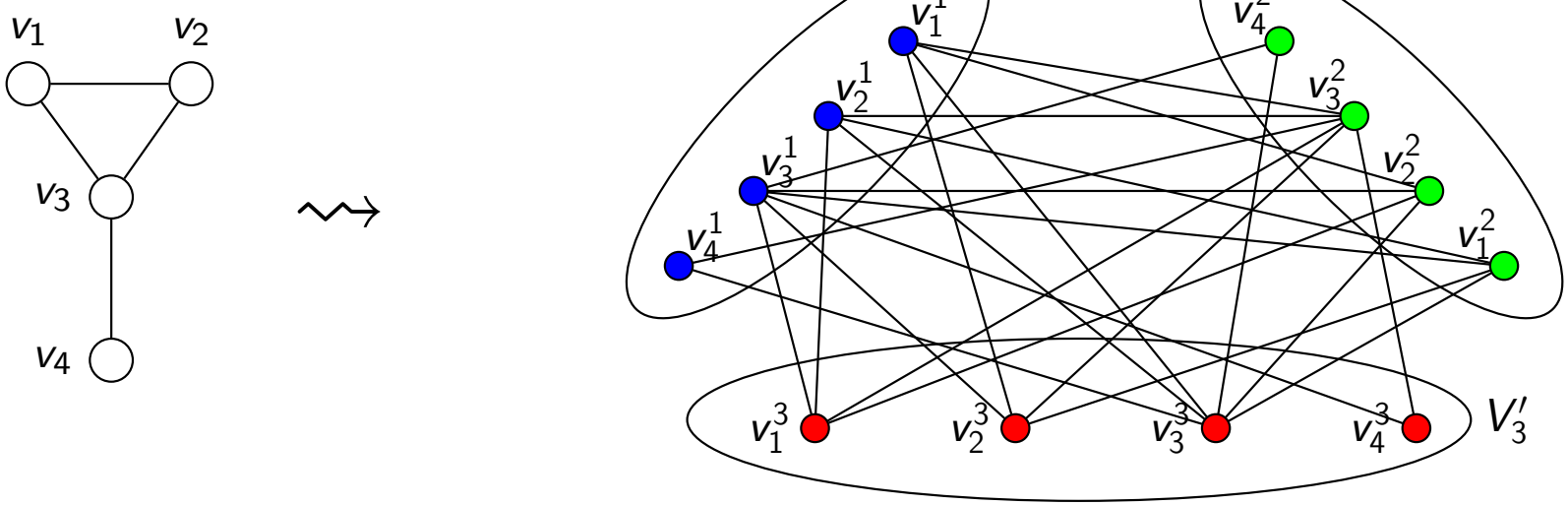

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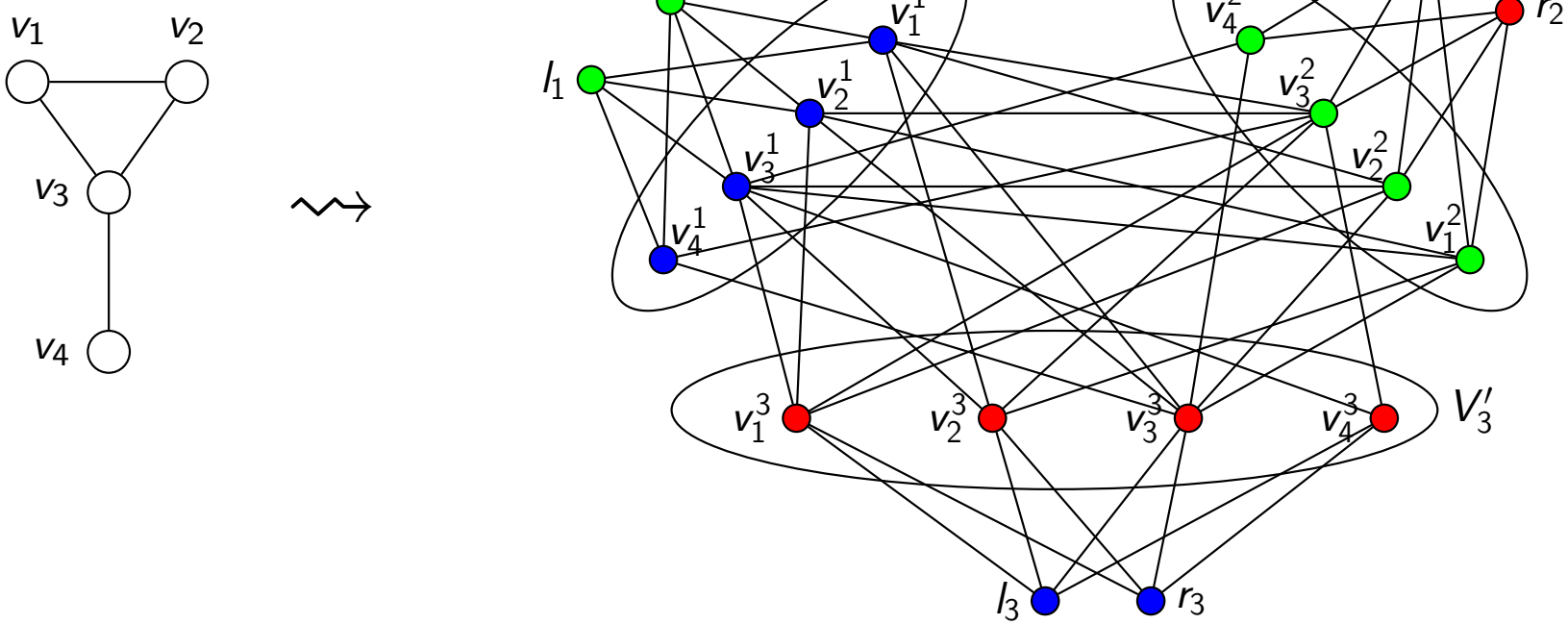

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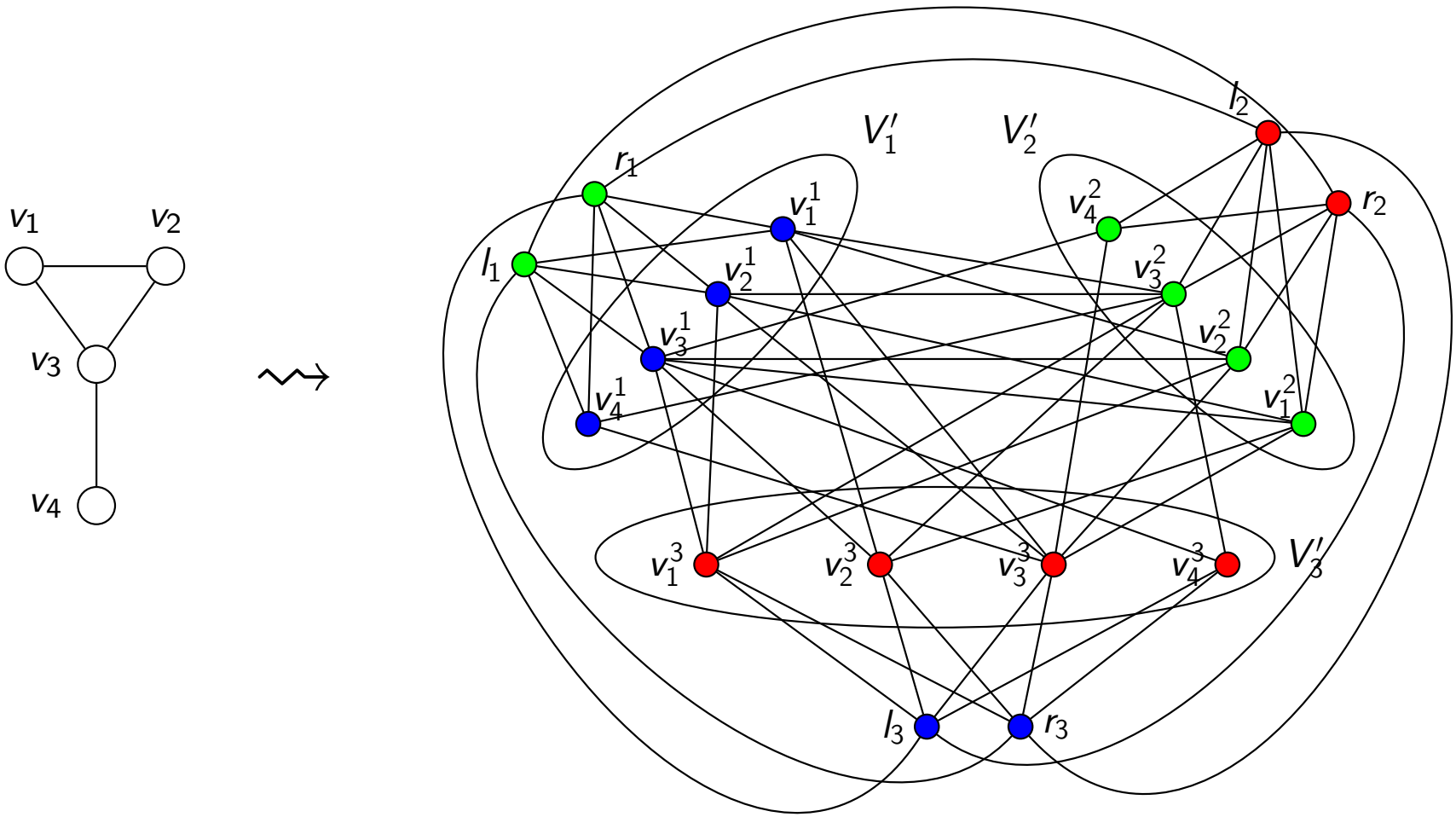

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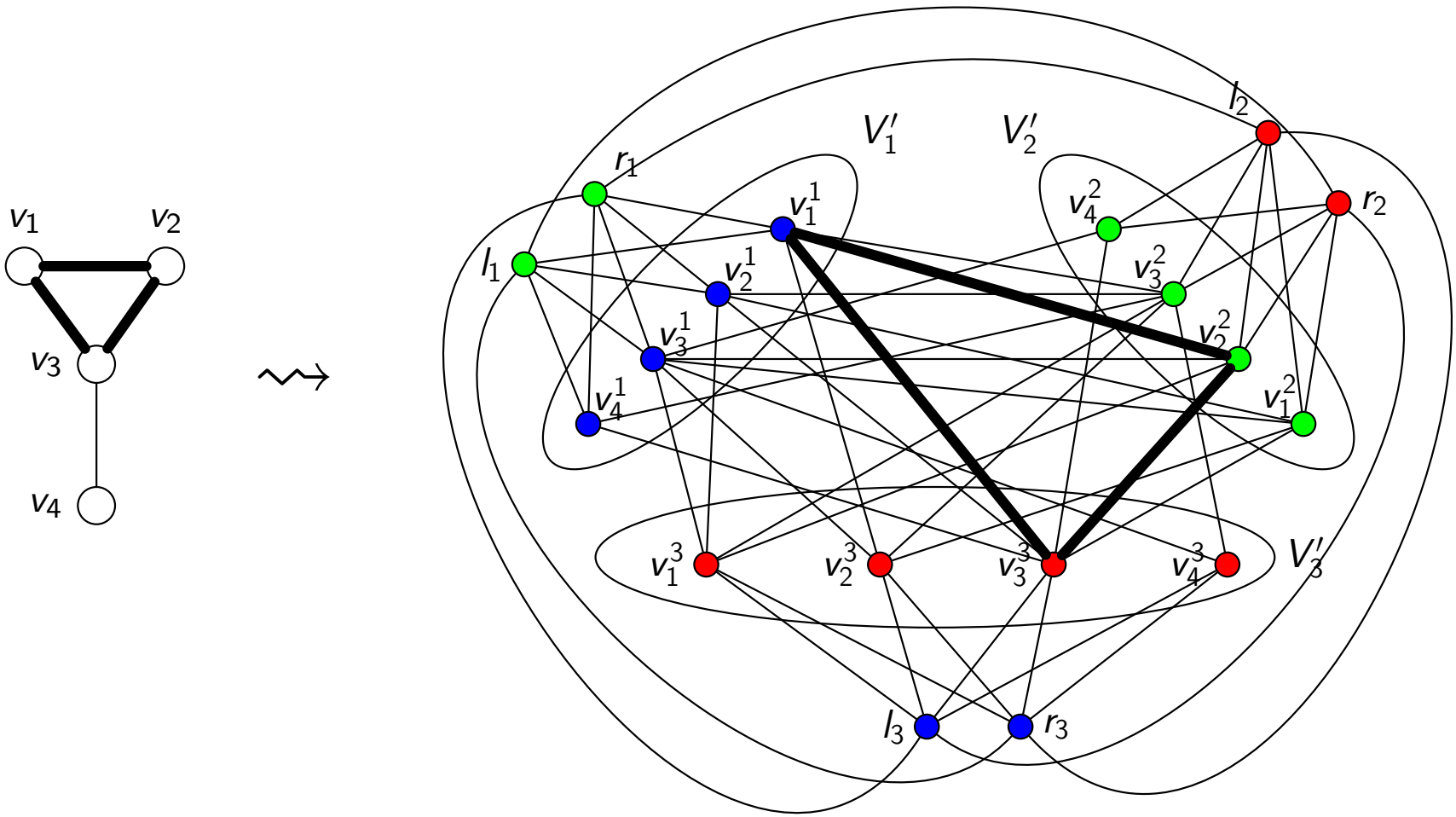

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Thank you

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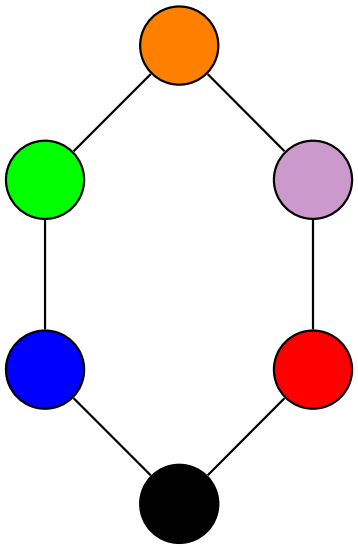

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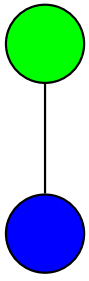

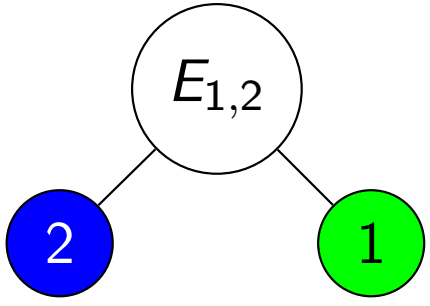

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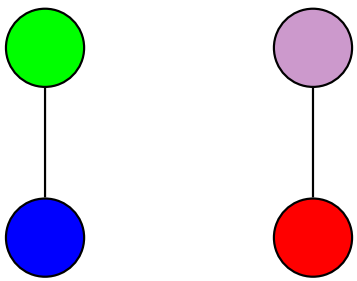
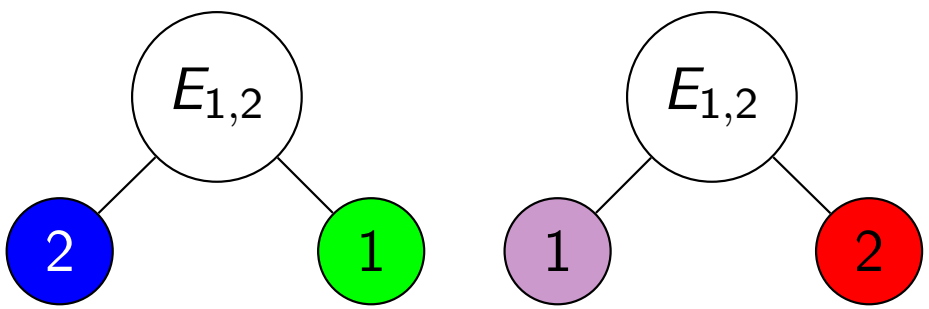
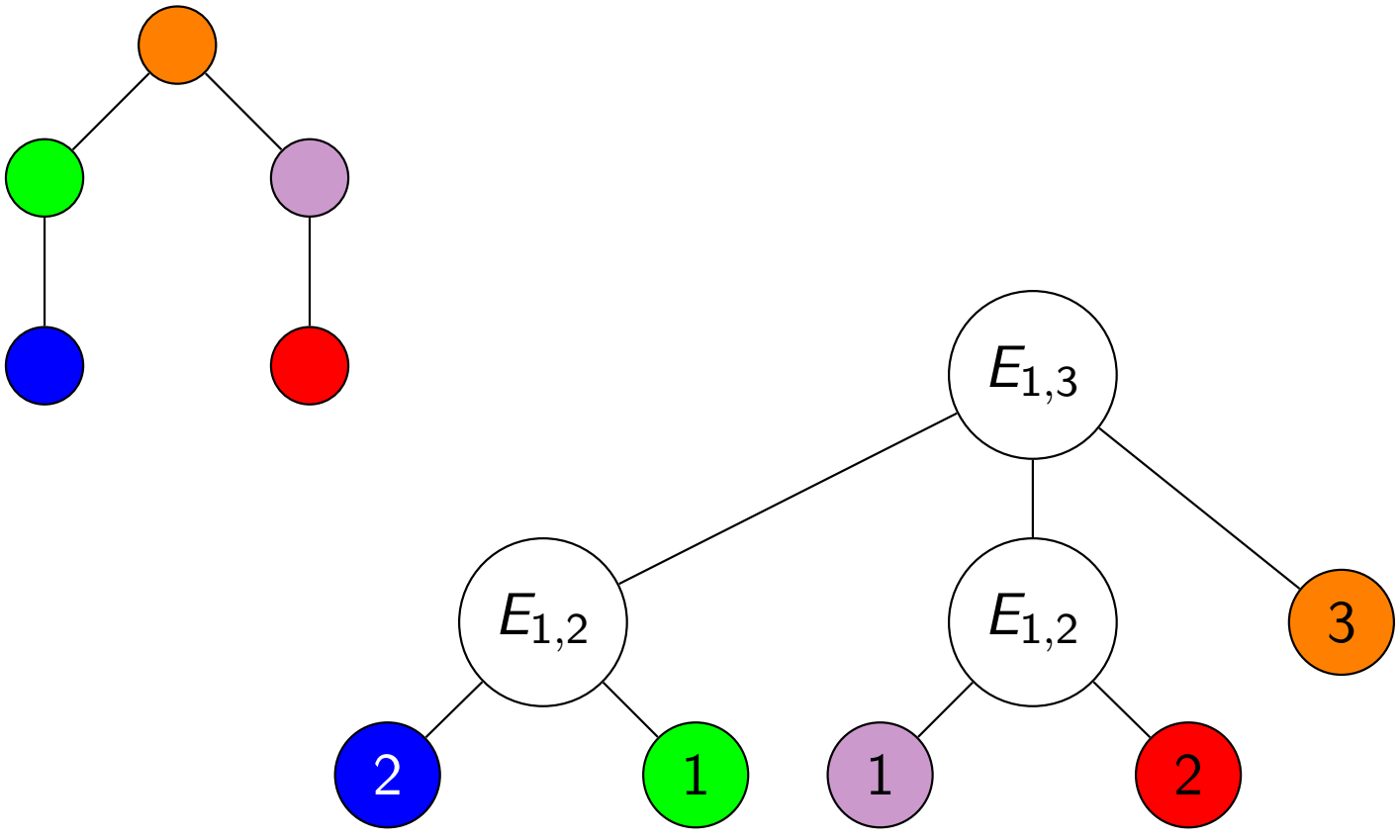




Clique-Width I



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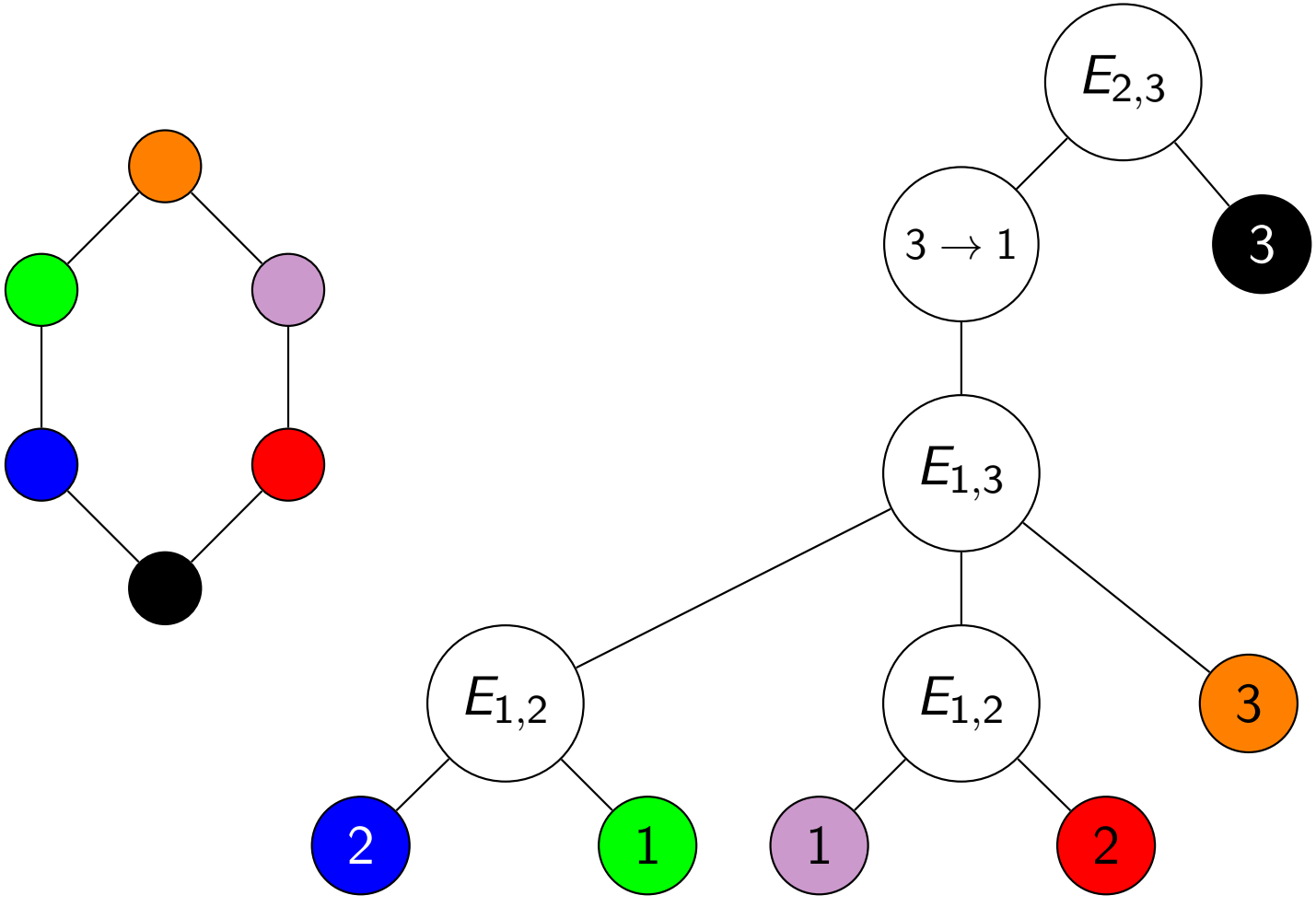

Clique-Width II

Theorem

Triangle Enumeration *is solvable in* $O(n^2 + n \cdot k^2 + \#\mathsf{T})$ *time, given a* k*-expression of the input graph.*

Clique-Width II

Given a k-expression tree B of G one can compute every node u in B the partition of V_u into its at most k twin-classes where V_u is the set of vertices corresponding to the leaves of the subtree of B rooted at u. [Bui-Xuan et al.: European Journal of Combinatorics, 2013]

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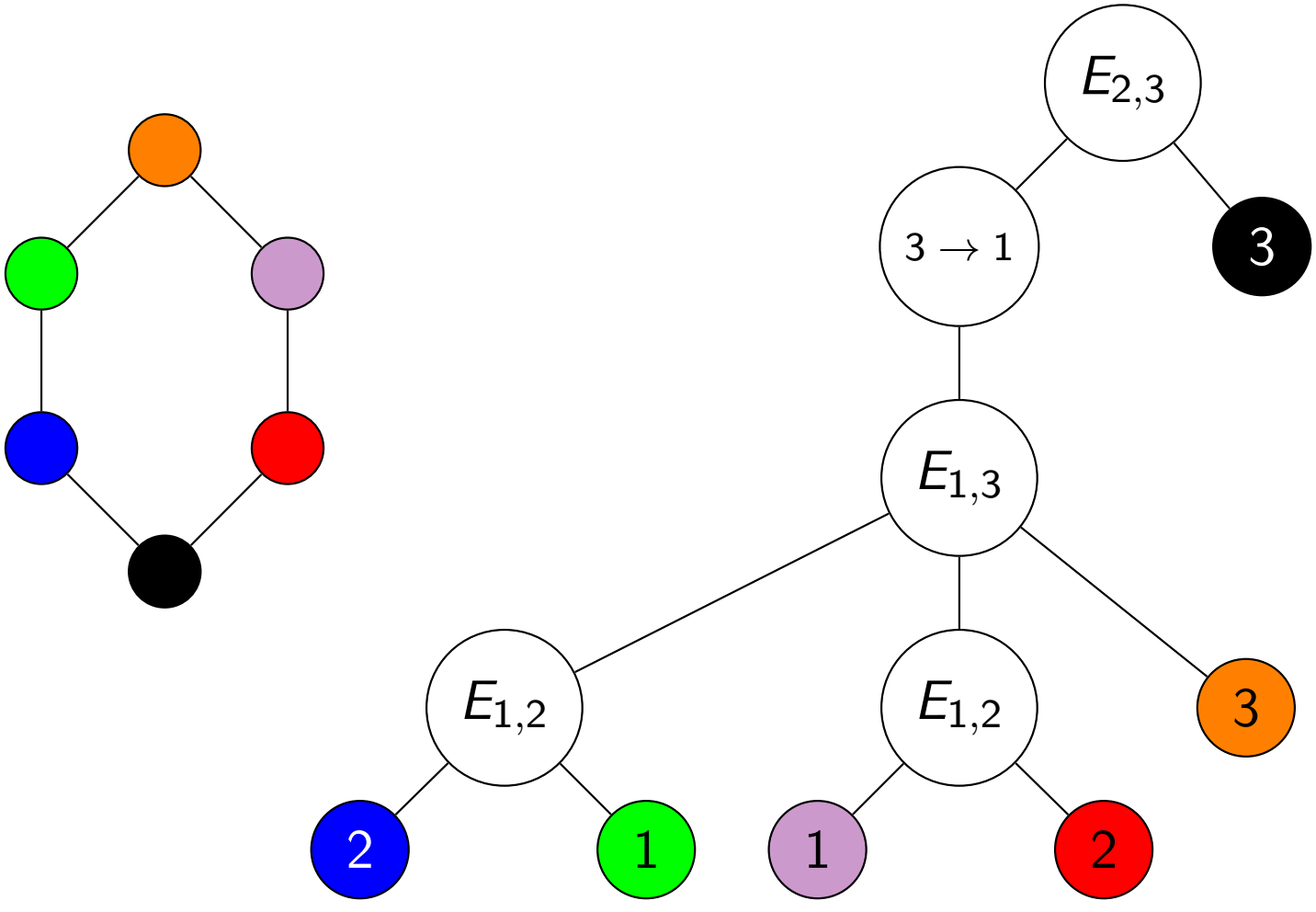
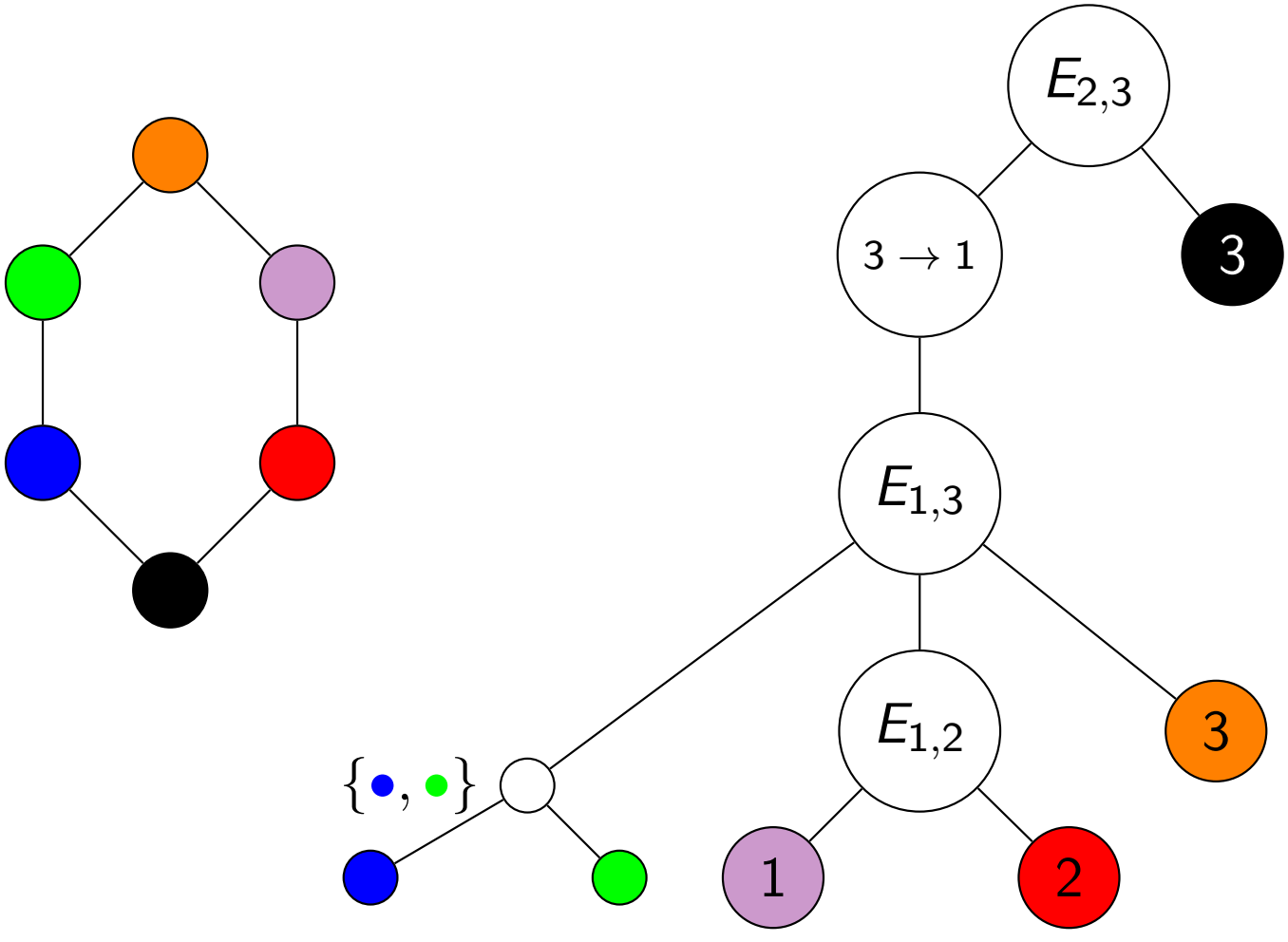
Definition (twin-class)

A **twin-class** in V_u is a set $V' \subseteq V_u$ of vertices such that every vertex in $V \setminus V_u$ either has all vertices in V' as its neighbors or none of them.

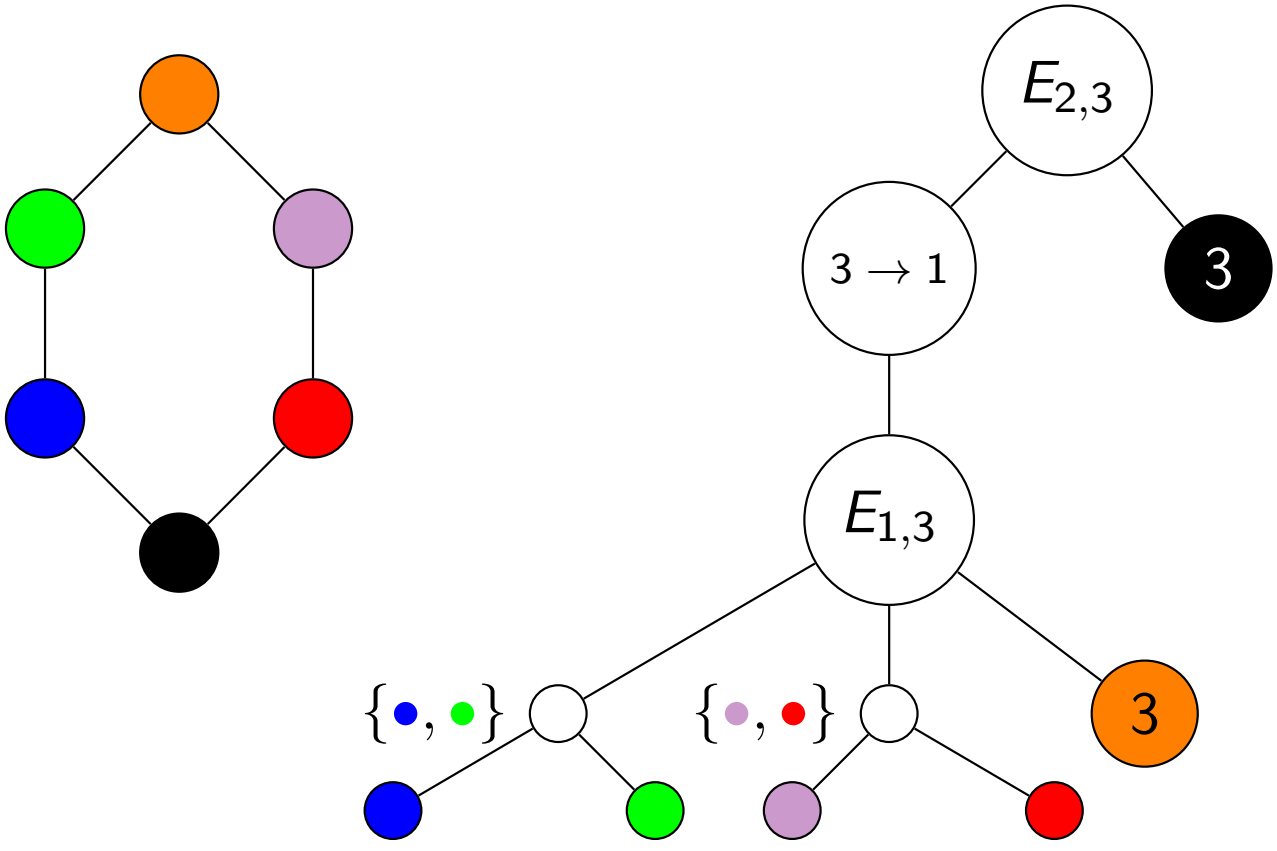
Clique-Width II



Clique-Width II



Clique-Width II



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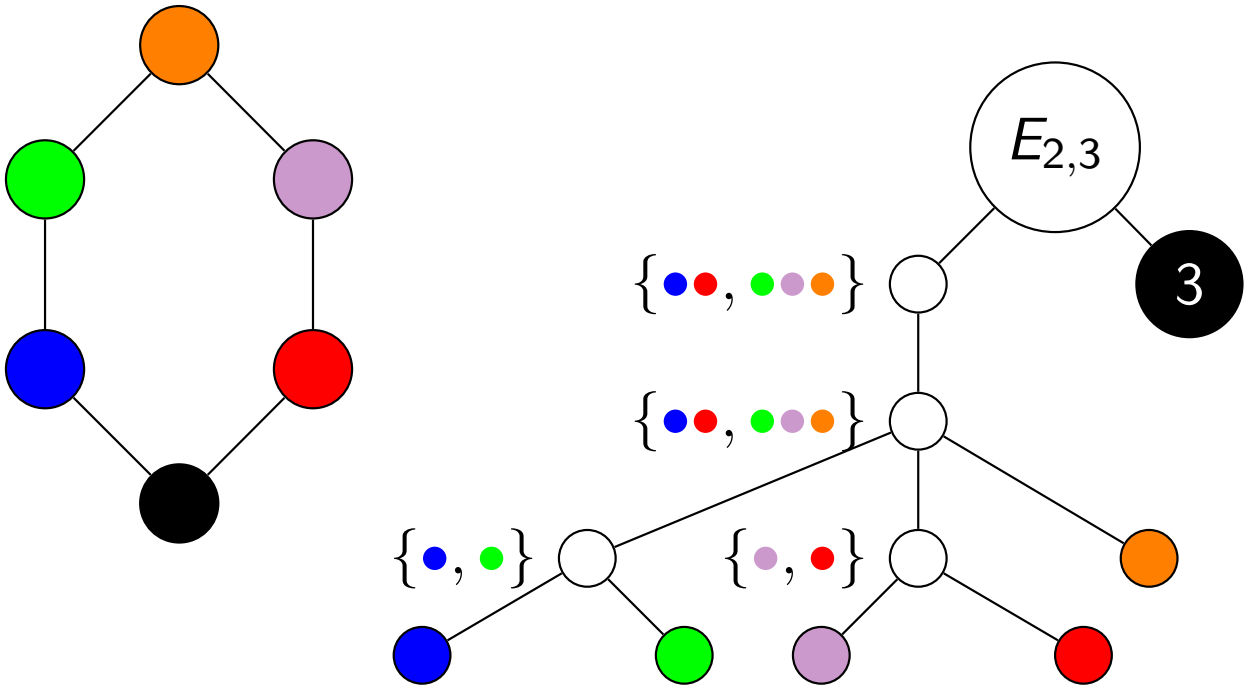

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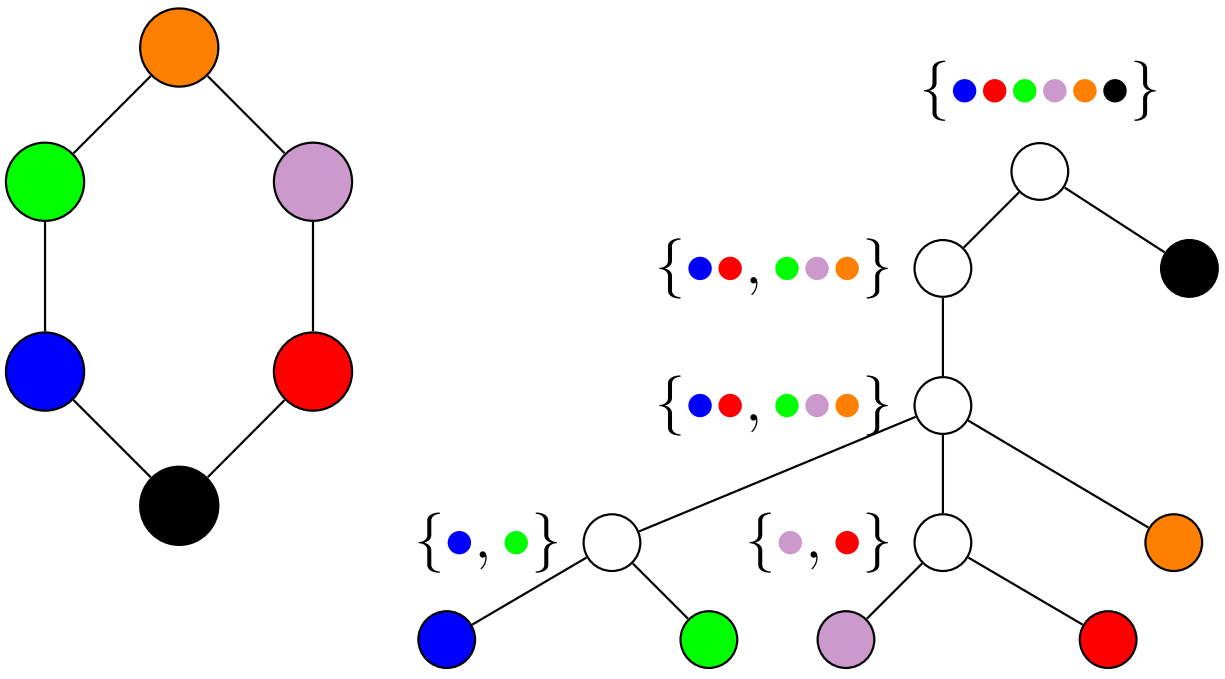

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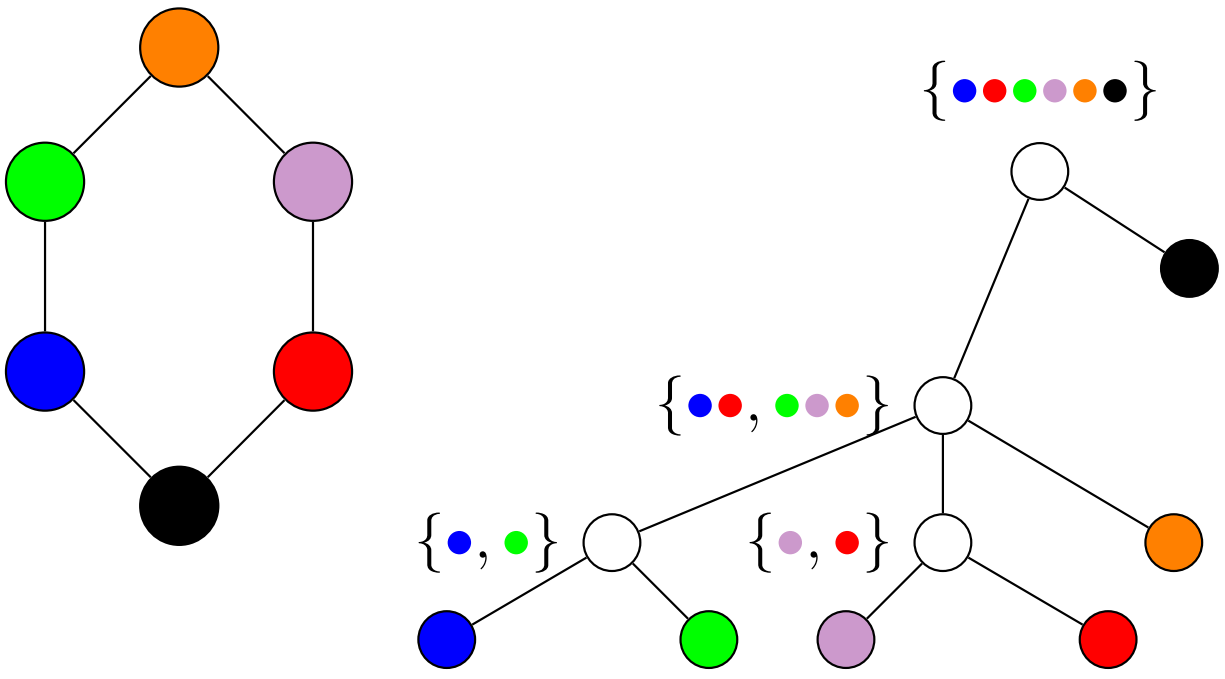

Clique-Width II

Properties of the tree of twin-classes (module decomposition):

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- Each twin-class is fully contained in one of the twin-classes of the parent node.
- The root node has only a single twin-class containing all vertices (of G).

Clique-Width III

u, v, w: nodes (where u, w are the children of v (if they exist))
i, j, l: twin-classes of v
v_i, v_j, v_l: arbitrary vertices in the twin-classes i, j, l, respectively
M_h^x: set of all twin-classes of x that are subsets of h

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Dynamic program:
$E_{i,j}^v$: all edges between vertices in i and j
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A leave node has no edges and no triangles.

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An inner node has

$$E_{i,j}^v = \bigcup_{o \in M_i^u} \bigcup_{p \in M_j^u} E_{o,p}^u \cup \bigcup_{o \in M_i^u} \bigcup_{p \in M_j^w} \{\{x, y\} \mid x \in o, y \in p, \{v_o, v_p\} \in E\}$$

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$$T_{i,j,l}^v = \bigcup_{o \in M_i^u} \bigcup_{p \in M_j^u} \bigcup_{q \in M_l^u} T_{o,p,q}^u \cup$$

$$\bigcup_{o \in M_i^u} \bigcup_{p \in M_j^u} \bigcup_{q \in M_l^w} \{\{x, y, z\} \mid x \in o, y \in p, z \in q, \{v_i, v_l\}, \{v_j, v_l\} \in E\}$$

Clique-Width III

- Each edge and triangle is computed exactly once.
- First step by Bui-Xuan et al. takes $O(n^2)$ time.
- Dynamic program takes $O(m + \#\mathsf{T} + n \cdot k^2)$ time.

Distance to d-Degenerate Graphs and Maximum Degree

Theorem

Triangle Enumeration *parameterized by distance to d-degenerate graphs and maximum degree Δ_D in a set D such that $G - D$ is d-degenerate is solvable in $O(|D| \cdot \Delta_D^2 + n \cdot d^2)$ time provided that the set D is given.*

Distance to d-Degenerate Graphs and Maximum Degree

1. List all triangles which do not contain any vertices in D. To this end, compute the d-degenerate graph $G' = G - D$ and list all triangles contained in G' in $O(n \cdot d^2)$ time using the degeneracy order of G'.

Distance to d-Degenerate Graphs and Maximum Degree

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2. List all triangles with at least one vertex contained in D. For each $u \in D$, iterate over all of the at most Δ_D^2 possible pairs of neighbors $v, w \in N(u)$. For each pair $v, w \in N(u)$, check in constant time whether $\{u, v, w\}$ is a new triangle.

General Lemma for Distance to Π

Lemma

Triangle Enumeration *parametrized by deletion set* K *to* Π *is solvable in* $O(m \cdot |K| + n + x)$ *time if* Triangle Enumeration *on a graph in* Π *is solvable in* $O(x)$ *time.*

General Lemma for Distance to Π

Let K be a set of vertices such that $G' = G - K$ is a graph in Π. By assumption, all triangles within G' can be listed in $O(x)$ time.

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- Read the whole input and fix an arbitrary order $\leq_a$ of the vertices in K.
- Check for each edge $\{u, w\}$ and each vertex $v \in K$ whether $\{u, v, w\}$ is a triangle and for all $x \in \{u, w\} \cap K$ it holds that $v \leq_a x$. If both conditions hold, then list $\{u, v, w\}$ as a new triangle.

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- Check for each edge $\{u, w\}$ and each vertex $v \in K$ whether $\{u, v, w\}$ is a triangle and for all $x \in \{u, w\} \cap K$ it holds that $v \leq_a x$. If both conditions hold, then list $\{u, v, w\}$ as a new triangle.

Since $v \in K$ holds, this algorithm does not list any triangles which do not contain vertices in K and it lists all triangles with at least one vertex in K exactly once.

Distance to Cographs

Theorem

Triangle Enumeration *parametrized by deletion set K to cographs is solvable in $O(\#\mathsf{T} + n + m \cdot |K|)$ time.*

Distance to Cographs

Each cograph has a binary cotree representation that can be computed in linear time.

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A cotree is a tree with V as a set of leaves and inner vertices that are either labeled with union or join.
A union of two graphs is disjoint and a join of two graphs is the same vertices and edges plus all edges between the two sets of vertices.

Distance to Cographs

Dynamic program that stores for each node in the cotree all vertices $V(p)$, all edges $E(p)$ and all triangles $T(p)$ in the corresponding subgraph of G.

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Let q_1, q_2 be the children of an inner node p in the cotree.

- A single leaf node has one vertex and no edges or triangles.
- A union node has vertices $V(q_1) \cup V(q_2)$, edges $E(q_1) \cup E(q_2)$ and triangles $T(q_1) \cup T(q_2)$.

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- A join node has

$$V(p) = V(q_1) \cup V(q_2),$$
$$E(p) = E(q_1) \cup E(q_2) \cup \{\{x, y\} \mid x \in V(q_1) \wedge y \in V(q_2)\} \text{ and}$$
$$T(p) = T(q_1) \cup T(q_2) \cup \{\{x, y, z\} \mid x \in V(q_1) \wedge \{y, z\} \in E(q_2)\} \cup$$
$$\{\{x, y, z\} \mid x \in V(q_2) \wedge \{y, z\} \in E(q_1)\}.$$

Distance to Cographs

- There are n leaf-nodes in the cotree each of which require a constant amount of time to compute.
- There are at most $n - 1$ union nodes each of which only require a constant amount of time as they only need to point on their children's values.

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The global running time of this algorithm is in $O(\#\mathsf{T}+n+m)$.

Enum-Advice Kernelization $\Longrightarrow$ Solving Algorithm

Lemma

Let $\mathcal{R}$ be an enum-advice kernelization of a parameterized enumeration problem P such that for every instance (x, k) of P:

- *$\mathcal{R}$ runs in $O((|x| + k)^c)$ time for some constant c;*
- *the unparameterized version of P can be solved in $g(|x|)$ time;*
- *the kernelization computes the pair (I, A) where $|I| \leq h(k)$, and $\mathcal{T}_f$ takes $O(|I|^d)$ time between generating two solutions for some constant d;*
- *$\#s$ denotes the number of solutions in I and $\#S$ denotes the number of solutions in x.*

Then, P can be solved in $O((|x| + k)^c + g(h(k)) + (\#s + \#S) \cdot h(k)^d)$ time.

Enum-Advice Kernelization $\Longrightarrow$ Solving Algorithm

1. Compute the kernel (I, A) in $O((|x| + k)^c)$ time.
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3. Compute all solutions of I in $O(g(|I|) \subseteq O(g(h(k)))$ time.
4. Apply $\mathcal{T}_f$ on all solutions of I.
   This takes $O((\#\mathsf{s} + \#\mathsf{S}) \cdot |I|^d) \subseteq O((\#\mathsf{s} + \#\mathsf{S}) \cdot h(k)^d)$ time.

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This algorithm takes $O((|x| + k)^c + g(h(k)) + (\#\mathsf{s} + \#\mathsf{S}) \cdot h(k)^d)$ time and lists all triangles in x.

Feedback Edge Number I

Lemma

Let $G = (V, E)$ be an undirected graph and let F be a feedback edge set in G. All triangles $\{u, v, w\}$ where at least one of the edges between the three vertices is not in F can be enumerated in $O(n + m)$ time. There are at most $2|F|$ such triangles.

Feedback Edge Number I

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  - If $p(u) \neq p(v)$ and $\{p(u), v\} \in E$, then we list $\{u, v, p(u)\}$ as a triangle.
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Note that for every edge in F we list at most two triangles.

Feedback Edge Number II

Theorem

Triangle Enumeration *parameterized by feedback edge number k admits a constant-delay enum-advice kernel with at most $2k + 3$ vertices and $k + 3$ edges which can be computed in $O(n + m)$ time.*

Feedback Edge Number II

For every edge $e \in F$ put e and both of its endpoints into the kernel graph. Compute the feedback edge number $k' \leq k$.

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For every edge $e \in F$ put e and both of its endpoints into the kernel graph. Compute the feedback edge number $k' \leq k$. Compute all triangles in G with at least one edge in $E \setminus F$ and set A to be the set of all triangles found.

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For every edge $e \in F$ put e and both of its endpoints into the kernel graph. Compute the feedback edge number $k' \leq k$.
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If $A \neq \emptyset$, then add one extra triangle $\{x, y, z\}$ where $x, y, z \notin V$.

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$$f(w, A) = \begin{cases} A & \text{if } w = \{x, y, z\}, \\ \{w\} & \text{else.} \end{cases}$$

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- For two solutions p, q in I that are not $\{x, y, z\}$, if $p \neq q$, then $f(p, A) \cap f(q, A) = \{p\} \cap \{q\} = \emptyset$.

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- For two solutions p, q in I that are not $\{x, y, z\}$, if $p \neq q$, then $f(p, A) \cap f(q, A) = \{p\} \cap \{q\} = \emptyset$.
- By construction, $f(\{x, y, z\}, A)$ contains all triangles in G where at least one of the edges is not in F. Since all edges in F are included in G_I, all other triangles are contained in G_I.

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Each vertex or one of its twins is contained in the kernel instance.

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- $k' = k$

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- $\bigcup_{w:\mathtt{Sol}(I',k')\setminus\{\{a,b,c\}\}} f(w, A)$: all triangles with at least two vertices in D.

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Twins of twins are twins.

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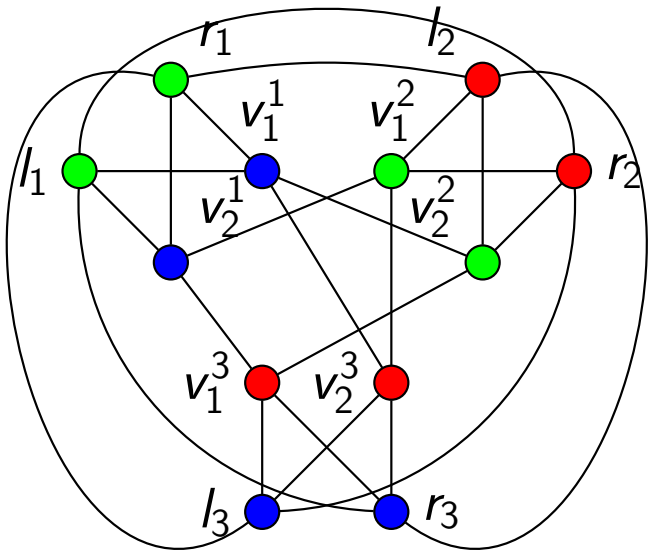
  - deleting all but one twin in linear time
- f: constant delay time (trivial lookup)

One Reduction to Rule Them All (Proof)

1. $k' \leq 9$



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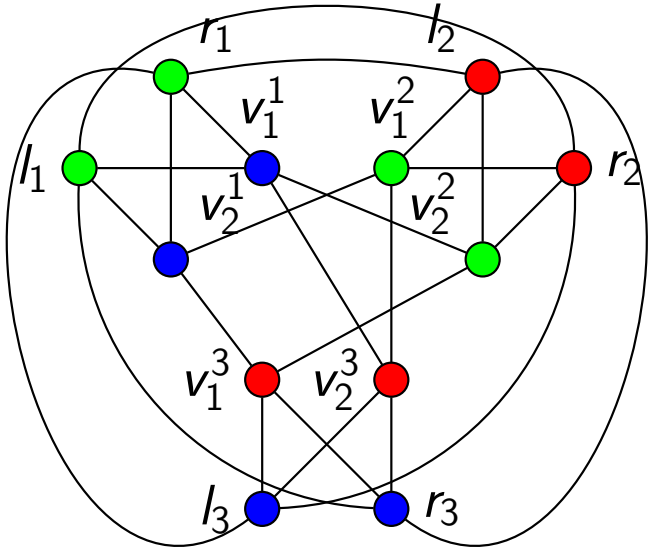

- chromatic number

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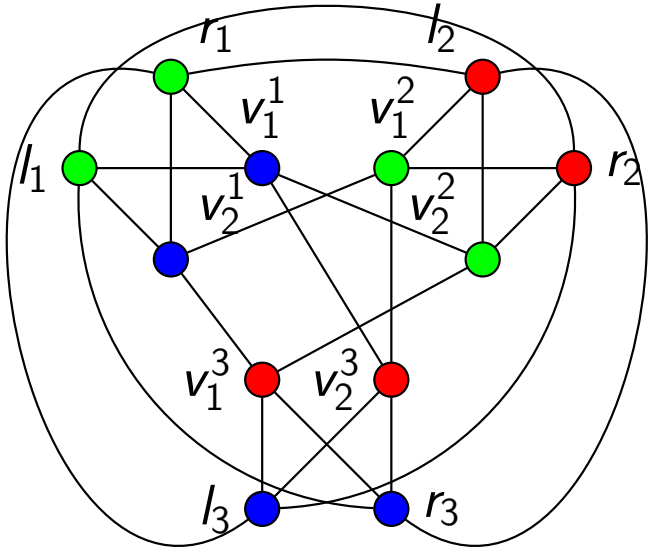

- chromatic number
- domination number (l_1, l_2, l_3)

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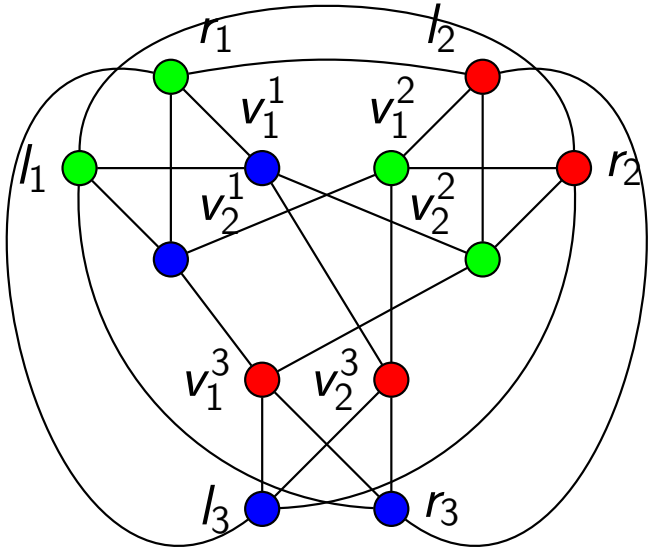

- chromatic number
- domination number (l_1, l_2, l_3)
- diameter

One Reduction to Rule Them All (Proof)

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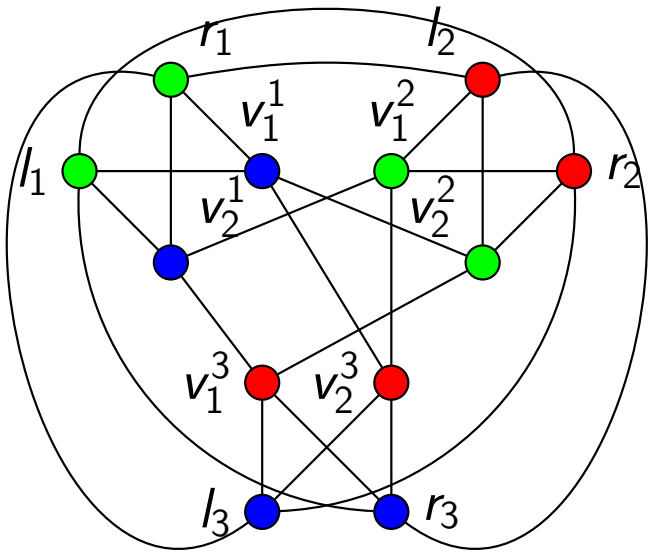

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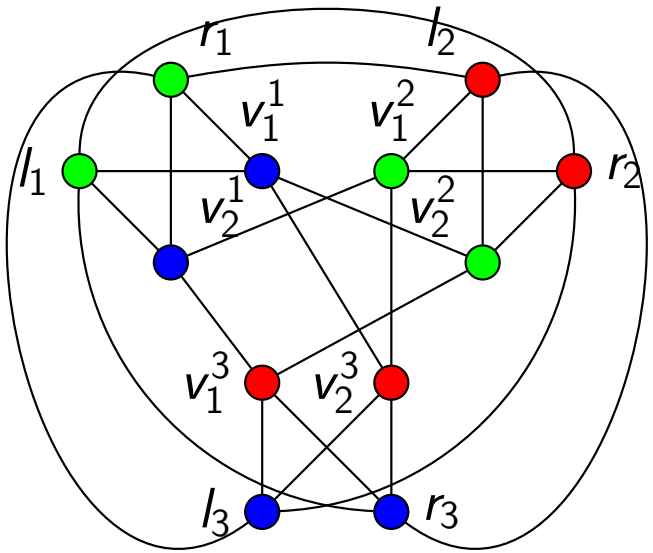
- l_i, r_i are not part of any triangle.



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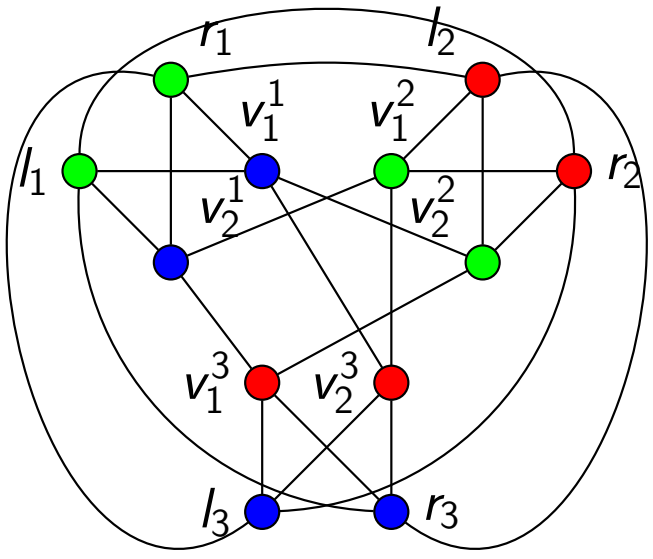
- Each triangle has to contain one red, one blue and one green vertex.



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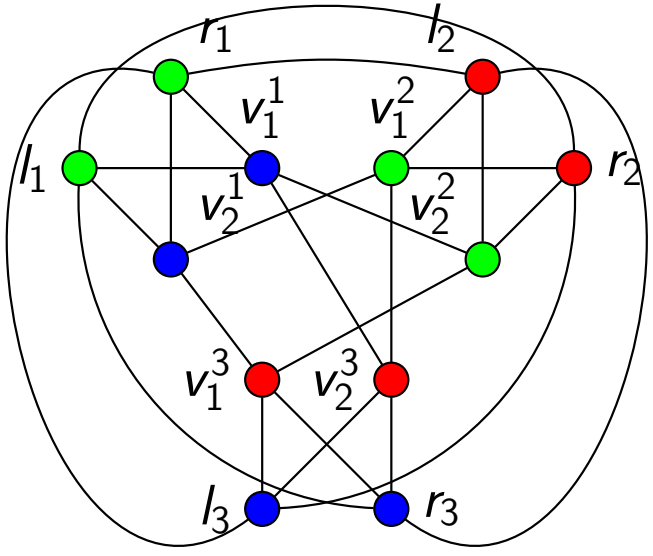

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- $(v_i^1, v_j^2, v_l^3) \Leftrightarrow (v_i, v_j, v_l)$

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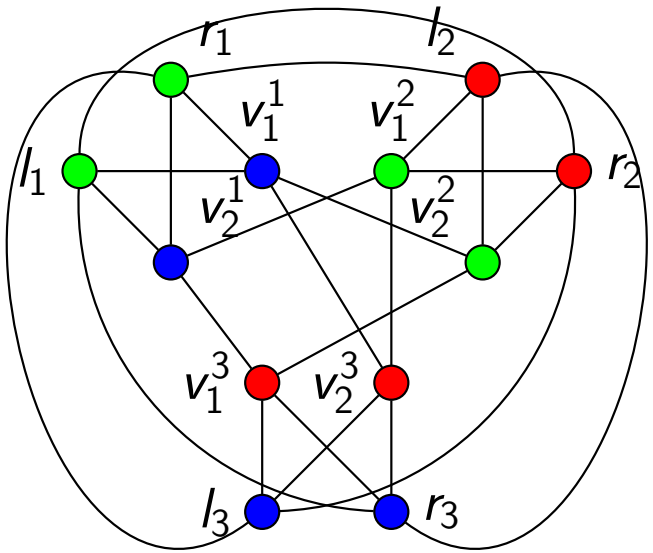
3. $|I'| \in O(|I|)$



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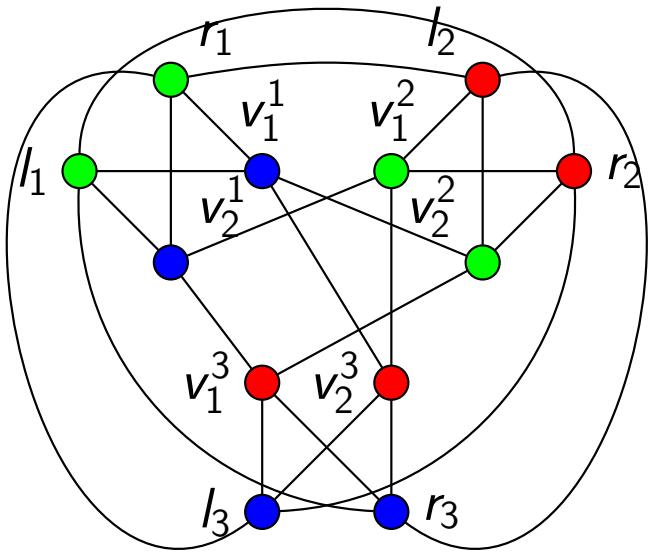
$n' = 3n + 6$
$m' = 6m + 6n + 6$



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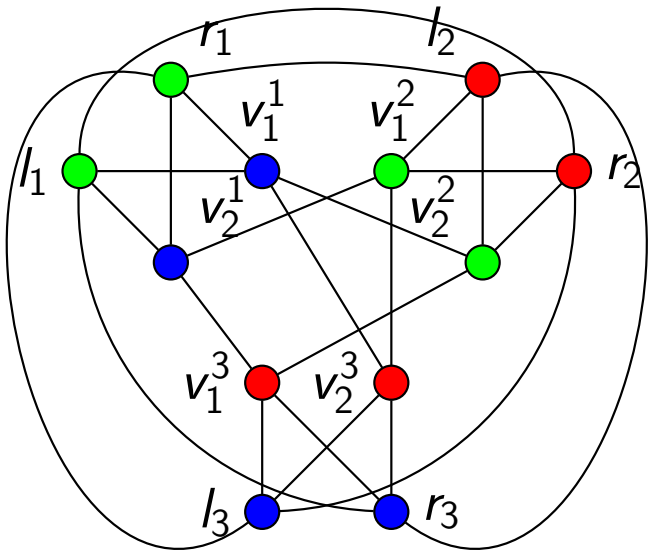

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Each operation takes constant time.