

**GENERALIZED
SATISFIABILITY
PROBLEMS
VIA
OPERATOR
ASSIGNMENTS**

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Talk plan

- I. Background and motivation
- II. Problem statement
- III. Results

Part I

BACKGROUND AND MOTIVATION

Systems of Polynomial Equations over Rings

Variables:

$$X_1, \dots, X_n$$

Systems of polynomial equations (arithmetic in a ring):

$$\begin{aligned} X_2 X_3 - 1 &= 0 \\ X_1 X_2 X_3 - 2X_1 X_3 + X_2 &= 0 \\ X_3 + X_4 - 2X_1 + 1 &= 0 \\ X_1 + X_2 + X_3 - 2 &= 0 \end{aligned}$$

An important example: Mermin-Peres Magic Square

Nine variables, fifteen equations:

$$X_{11} X_{12} X_{13} = +1$$

$$X_{21} X_{22} X_{23} = +1$$

$$X_{31} X_{32} X_{33} = +1$$

$$X_{11} X_{21} X_{31} = +1$$

$$X_{12} X_{22} X_{32} = +1$$

$$X_{13} X_{23} X_{33} = -1$$

$$X_{11}^2 = X_{12}^2 = \dots = X_{33}^2 = +1$$

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Proof of unsatisfiability over commutative rings (e.g., \mathbb{C})

$$X_{11}X_{12}X_{13}X_{21}X_{22}X_{23}X_{31}X_{32}X_{33} = +1$$

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Remarks:

- ▶ Do not even need $X_{ij}^2 = +1$.
- ▶ Relies heavily on the fact that **product commutes**.

Indeed ...

There is a solution in 4x4 complex matrices

$$\begin{array}{rcccl} I \otimes Z & Z \otimes I & Z \otimes Z & = & +I \\ X \otimes I & I \otimes X & X \otimes X & = & +I \\ X \otimes Z & Z \otimes X & Y \otimes Y & = & +I \\ \parallel & \parallel & \parallel & & \\ +I & +I & -I & & \end{array}$$

where X, Y, Z are the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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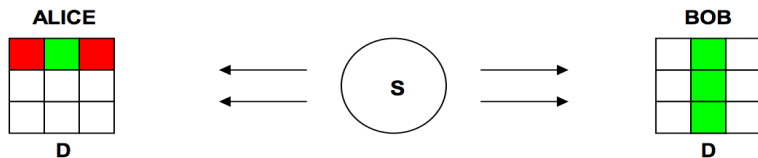
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Note:

$$(I \otimes Z)^2 = (X \otimes I)^2 = \dots (Y \otimes Y)^2 = +I$$

Where does this come from? Quantum entanglement

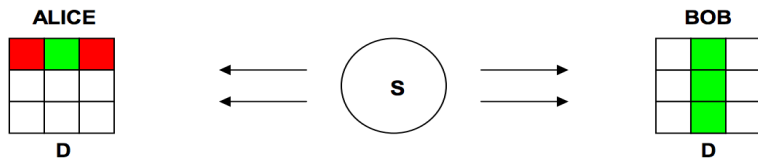
[Einstein-Podolsky-Rosen 1935], [Bell 1964], [Mermin 1990]



$\hat{p}_{ij,ab} :=$ “empirical probability that ij lights as ab ”

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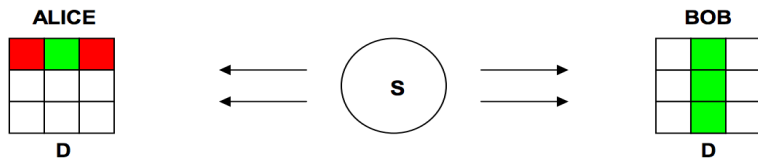
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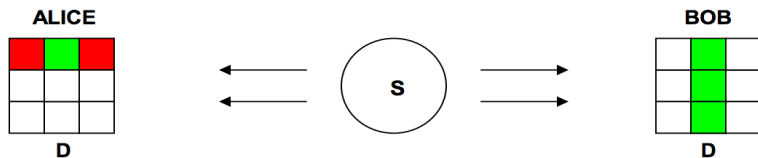
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$\hat{p}_{ij,ab} :=$ “empirical probability that ij lights as ab ”

- ▶ **Cannot be explained** by a **classical probability distribution**;
- ▶ **Can be explained** by **quantum entanglement**;

Back to equations [Cleve and Mittal, ICALP 2014]

Variables ranging over Hermitian matrices

$$\begin{array}{l} X_{11} X_{12} X_{13} \\ X_{21} X_{22} X_{23} \\ X_{31} X_{32} X_{33} \end{array}$$

Back to equations [Cleve and Mittal, ICALP 2014]

Variables ranging over Hermitian matrices
Equations enforcing unitaries

$$\begin{array}{ccc} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{array}$$

$$X_{ij}^2 = +1 \quad \text{for all } i \text{ and } j$$

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Variables ranging over Hermitian matrices

Equations enforcing unitaries

Equations enforcing constraints

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Back to equations [Cleve and Mittal, ICALP 2014]

Variables ranging over Hermitian matrices

Equations enforcing unitaries

Equations enforcing constraints

Equations enforcing joint measurability

$$X_{11} X_{12} X_{13} = +1$$

$$X_{21} X_{22} X_{23} = +1$$

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$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ \perp & \perp & \perp \\ \hline \perp & \perp & \perp \end{array}$$

$$X_{ij}^2 = +1 \quad \text{for all } i \text{ and } j$$

$$X_{ij} X_{kl} = X_{kl} X_{ij} \quad \text{if } i = k \text{ or } j = l$$

Part II

PROBLEM STATEMENT

Informal statement

Characterize the types of constraints that allow such *gedanken* experiments that are quantum realizable **but not** classically realizable.

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Characterize the types of constraints that allow such *gedanken* experiments that are quantum realizable **but not** classically realizable.

E.g.:

parity constraints of arity ≥ 3 do,
but parity constraints of arity ≤ 2 don't.

Schaefer's framework for generalized satisfiability

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Boolean domain: $\{\pm 1\}$ with $+1 = \textit{false}$ and $-1 = \textit{true}$;

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$$R : \{\pm 1\}^r \rightarrow \{\pm 1\}$$

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Fourier-Welsh transform
 $R(X_1, \dots, X_r) = -1$

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Examples:

OR	disjunctions of literals
LIN	linear equations over \mathbb{Z}_2
1-IN-3	triples with one -1 and two $+1$ components
NAE	triples with not-all-equal components


Generalized Satisfiability Problems: SAT(A)

$$\exists X_1 \cdots X_n (C_1 \wedge \cdots \wedge C_m)$$

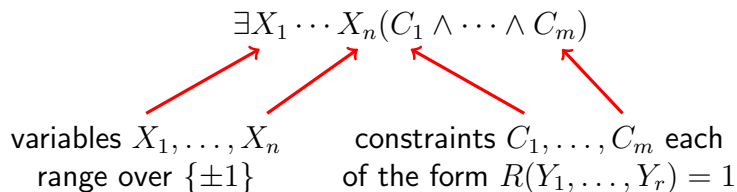
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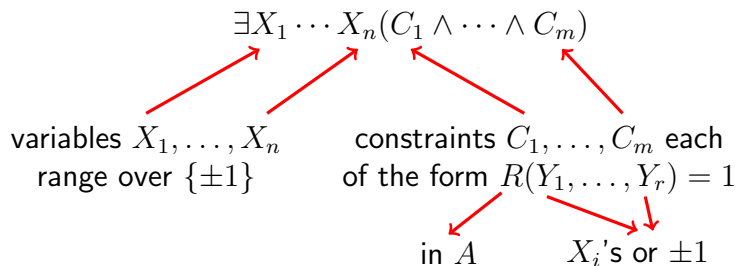
variables X_1, \dots, X_n
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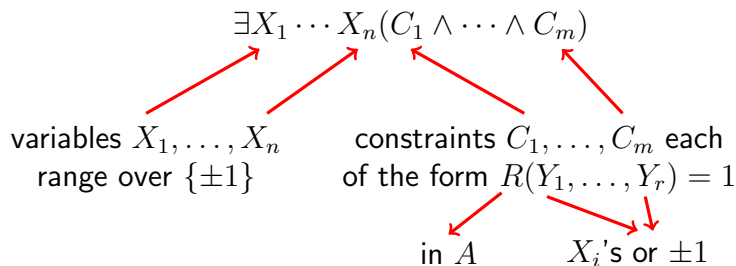
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Generalized Satisfiability Problems: SAT(A)



Examples:

3-SAT

HORN-SAT

LIN-SAT

1-IN-3-SAT

NAE-SAT

...

[Schaefer 1978]

... via Operator Assignments

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constraints C_1, \dots, C_m each
of the form $R(Y_1, \dots, Y_r) = I$
 $Y_i Y_j = Y_j Y_i$ for all $i, j \in [r]$

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and
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$$X_i^2 = I \text{ for all } i \in [n]$$

SAT(A)

satisfiability over Boolean domain

SAT*(A)

satisfiability over some finite-dimensional \mathcal{H}

SAT**(A)

satisfiability over some arbitrary \mathcal{H}

Gap Instances

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Mermin-Peres Magic Square

Unsatisfiable SAT-instance of LIN

Satisfiable SAT*-instance of LIN

a SAT-vs-SAT* gap for LIN

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Unsatisfiable SAT-instance of LIN

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a SAT-vs-SAT* gap for LIN

SAT-vs-SAT*

gap of the **first** kind

SAT-vs-SAT**

gap of the **second** kind

SAT*-vs-SAT**

gap of the **third** kind

Gap Instances

Mermin-Peres Magic Square

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Unsatisfiable SAT-instance of LIN
Satisfiable SAT*-instance of LIN

↓
a SAT-vs-SAT* gap for LIN

SAT-vs-SAT*	gap of the first kind
SAT-vs-SAT**	gap of the second kind
SAT*-vs-SAT**	gap of the third kind

Gaps of first kind for LIN exist [Mermin 1990]

Gaps of third kind for LIN exist [Slofstra 2017]

Gaps of **first kind** for 2-SAT or HORN do not exist [Ji 2014]

Part III

RESULTS

Classification

Theorem:

For every Boolean constraint language A ,

1. either gaps of every kind for A exist,
2. or gaps of no kind for A exist.

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Moreover:

gaps for A **do not** exist
iff
 A **is** of one of the following types:

- 0-valid
- 1-valid
- Horn
- dual Horn
- bijunctive

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iff
 A is of one of the following types:
iff
LIN is **not pp-definable** from A

$\left\{ \begin{array}{l} 0\text{-valid} \\ 1\text{-valid} \\ \text{Horn} \\ \text{dual Horn} \\ \text{bijunctive} \end{array} \right.$


Comparison with Schaefer's dichotomy for tractability

	Tractable	Gaps exist
0-valid/1-valid	YES	NO
Horn/dual-Horn	YES	NO
bijunctive	YES	NO
linear	YES	YES
anythingelse	NO	YES

Primitive Positive Definitions


$$R(Y_1, \dots, Y_r) \equiv \exists Z_1 \dots \exists Z_s (C_1 \wedge \dots \wedge C_t)$$

auxiliary variables constraints on the Y 's and Z 's



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Example:

$$\text{NAE}(X, Y, Z) \equiv (X \vee Y \vee Z) \wedge (\bar{X} \vee \bar{Y} \vee \bar{Z})$$

Proof technique

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Ingredient 1: gap preserving reductions

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If A is pp-definable from B ,
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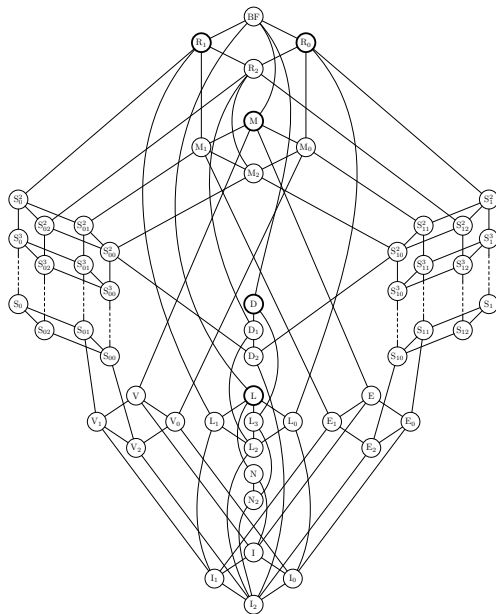
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Ingredient 2: Post's Lattice of Boolean co-clones

Theorem [Post 1941]:

There are countably many Boolean constraint languages up to pp-definability, **and we know them.**

Post's Lattice



More on Primitive Positive Definability

$$R(Y_1, \dots, Y_r) \equiv \exists Z_1 \cdots \exists Z_s (C_1 \wedge \cdots \wedge C_t)$$

pp-def Z_i 's range over $B(\mathbb{C})$ (i.e., over $\{\pm 1\}$ by $Z_i^2 = I$)

pp*-def Z_i 's range over $B(\mathcal{H})$, for some finite-dim \mathcal{H}

pp-def** Z_i 's range over $B(\mathcal{H})$, for some arbitrary \mathcal{H}

A Conservativity Theorem

Theorem:

For every two constraint languages A and B , the following statements are equivalent.

1. every relation in A is **pp-definable** from B
2. every relation in A is **pp*-definable** from B

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Corollary: OR is **not** pp*-definable from LIN

Closure Operations via Operators

R is **invariant** under $F : B(\mathcal{H}_1) \times \cdots \times B(\mathcal{H}_s) \rightarrow B(\mathcal{H})$ if

$$R(A_{1,1} \quad , \cdots , A_{1,r}) = I \text{ and commute}$$
$$\vdots \quad \ddots \quad \vdots$$

$$R(A_{s,1} \quad , \cdots , A_{s,r}) = I \text{ and commute}$$

$$R(F(\mathbf{A}_{*,1}), \cdots , F(\mathbf{A}_{*,r})) = I \text{ and commute}$$

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Lemma: If A is invariant under $F : \{\pm 1\}^s \rightarrow \{\pm 1\}$, then every $R \subseteq \{\pm 1\}^r$ pp*-definable from A is invariant under

$$F^*(X_1, \dots, X_s) := \sum_{S \subseteq [s]} \widehat{F}(S) \bigotimes_{i=1}^s X_i^{S(i)}$$

Proof by Example

Operator composition doesn't work:

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$$\parallel \quad \parallel \quad \parallel$$

$$\begin{array}{ccc} \perp & \perp & \perp \\ \hline \perp & \perp & \perp \end{array} \neq +1 (!)$$

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Operator composition doesn't work:

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Operator tensoring works:

$$\begin{aligned} (X_{11} \otimes X_{21} \otimes X_{31})(X_{12} \otimes X_{22} \otimes X_{32})(X_{13} \otimes X_{23} \otimes X_{33}) &= \\ (X_{11} X_{12} X_{13}) \otimes (X_{21} X_{22} X_{23}) \otimes (X_{31} X_{32} X_{33}) &= \\ (+I) \otimes (+I) \otimes (+I) &= \\ +I & \end{aligned}$$

Future Work

Question 1:

Is $\text{SAT}^*(\text{LIN})$ decidable?

(Note: Slofstra proved that $\text{SAT}^{**}(\text{LIN})$ is undecidable)

Question 2:

Is pp^{**} -definability = pp -definability also?

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