Expressive Power of Evolving Neural Networks Working on Infinite Input Streams

Olivier Finkel[1]
Joint work with Jérémie Cabessa[2]

[1] CNRS and University Paris 7
[2] Laboratoire d'Économie Mathématique, Université Paris 2

FCT 2017, Bordeaux 12 Septembre 2017

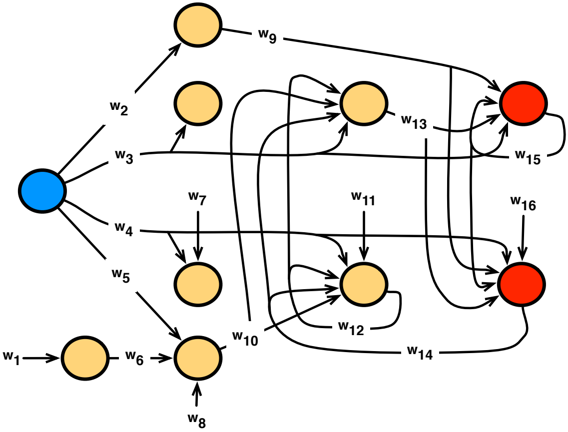
Introduction

- The computational capabilities of recurrent neural networks have mainly been studied in the context of classical computation: McCulloch & Pitts (1943), Turing (1948), Kleene (1956), von Neumann (1958), Minsky (1967), Papert (1969),. . . , Siegelmann & Sontag (1994-1995),. . .
- We provide a characterization of the computational power of recurrent neural networks in terms of their attractor dynamics, i.e., in the context of infinite input stream computation.

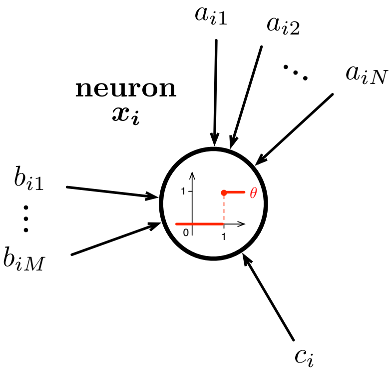
Introduction

- The computational capabilities of recurrent neural networks have mainly been studied in the context of classical computation: McCulloch & Pitts (1943), Turing (1948), Kleene (1956), von Neumann (1958), Minsky (1967), Papert (1969),. . . , Siegelmann & Sontag (1994-1995),. . .
- We provide a characterization of the computational power of recurrent neural networks in terms of their attractor dynamics, i.e., in the context of infinite input stream computation.

Recurrent Neural Network

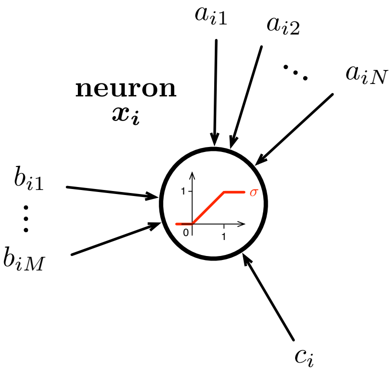


Boolean Neural Networks



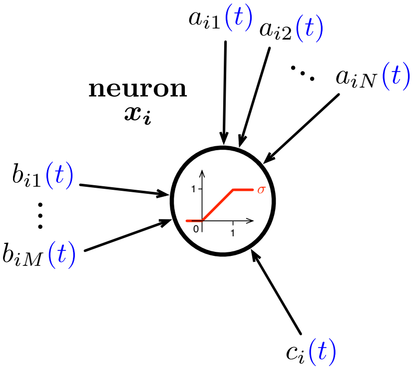
$$x_i(t+1) = \theta\left(\sum_{j=1}^{N} a_{ij}\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}\cdot u_j(t) + c_i\right)$$

Sigmoidal Neural Networks



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij} \cdot x_j(t) + \sum_{j=1}^{M} b_{ij} \cdot u_j(t) + c_i\right)$$

Sigmoidal Neural Networks



$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}(t)\cdot x_j(t) + \sum_{j=1}^{M} b_{ij}(t)\cdot u_j(t) + c_i(t)\right)$$

Neural Networks

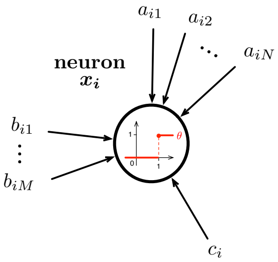
We consider three models of NNs:

1. Boolean rational NNs: B-NN[$\mathbb{Q}$]s
2. Sigmoidal static rational NNs: St-NN[$\mathbb{Q}$]s
3. Sigmoidal bi-valued evolving rational NNs: Ev$_2$-NN[$\mathbb{Q}$]s

Neural Networks

We consider three models of NNs:

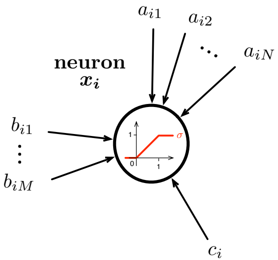
1. Boolean rational NNs: B-NN[$\mathbb{Q}$]s
2. Sigmoidal static rational NNs: St-NN[$\mathbb{Q}$]s
3. Sigmoidal bi-valued evolving rational NNs: Ev_2-NN[$\mathbb{Q}$]s



Neural Networks

We consider three models of NNs:

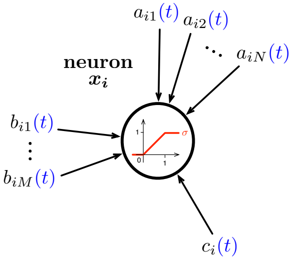
1. Boolean rational NNs: B-NN[ℚ]s
2. Sigmoidal static rational NNs: St-NN[ℚ]s
3. Sigmoidal bi-valued evolving rational NNs: Ev_2-NN[ℚ]s



Neural Networks

We consider three models of NNs:

1. Boolean rational NNs: B-NN[$\mathbb{Q}$]s
2. Sigmoidal static rational NNs: St-NN[$\mathbb{Q}$]s
3. Sigmoidal bi-valued evolving rational NNs: Ev$_2$-NN[$\mathbb{Q}$]s



Results (Classical Computation)

	Boolean	Static	Bi-valued Evolving
	FSA	TM	TM/poly(A)
	REG	**P**	**P/poly**
	Kleene 56 Minsky 67	Siegelmann & Sontag 95	Cabessa & Siegelmann 11,14

Muller Turing Machine

A *Muller Turing machine* consists of a classical TM with Muller acceptance condition. Muller table $\mathcal{T}$: collection of accepting sets of states.

- The ω-word u is *accepted* by $\mathcal{M}$ if there is an infinite run ρ_u of the machine $\mathcal{M}$ on u such that $\inf(\rho_u) \in \mathcal{T}$
- The ω-language *accepted* by $\mathcal{M}$ is the set of ω-words accepted by $\mathcal{M}$.

Complexity of ω-languages

The question naturally arises of the complexity of ω-languages accepted by various kinds of automata.

A way to study the complexity of ω-languages is to consider their topological complexity.

Topology on Σ^ω

The natural prefix metric on the set Σ^ω of ω-words over Σ is defined as follows:

For $u, v \in \Sigma^\omega$ and $u \neq v$ let

$$\delta(u, v) = 2^{-n}$$

where n is the least integer such that:

the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v.

This metric induces on Σ^ω the usual Cantor topology for which :

- *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$.
- *closed subsets* of Σ^ω are complements of *open subsets* of Σ^ω.

Borel Hierarchy

Σ^0_1 is the class of open subsets of Σ^ω,

Π^0_1 is the class of closed subsets of Σ^ω,

For any countable ordinal $\alpha \geq 2$:

Σ^0_α is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \Pi^0_\gamma$.

Π^0_α is the class of complements of Σ^0_α-sets

$\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$.

A set $X \subseteq \Sigma^\omega$ is a Borel set iff it is in $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$ where ω_1 is the first uncountable ordinal.

Borel Hierarchy

Below an arrow $\rightarrow$ represents a strict inclusion between Borel classes.

$$
\begin{array}{ccccccccccccccc}
 & & \Pi^0_1 & & & & & & & \Pi^0_\alpha & & & & \Pi^0_{\alpha+1} & \\
 & \nearrow & & \searrow & & \nearrow & & & \nearrow & & \searrow & & \nearrow & & \\
\Delta^0_1 & & & & \Delta^0_2 & & \cdots\ \cdots & \Delta^0_\alpha & & & & \Delta^0_{\alpha+1} & & & \cdots \\
 & \searrow & & \nearrow & & \searrow & & & \searrow & & \nearrow & & \searrow & & \\
 & & \Sigma^0_1 & & & & & & & \Sigma^0_\alpha & & & & \Sigma^0_{\alpha+1} &
\end{array}
$$

Beyond the Borel Hierarchy

There are some subsets of Σ^ω which are not Borel. Beyond the Borel hierarchy is the projective hierarchy.

The class of Borel subsets of Σ^ω is strictly included in the class Σ_1^1 of analytic sets which are obtained by projection of Borel sets.
A set $E \subseteq \Sigma^\omega$ is in the class Σ_1^1 iff :

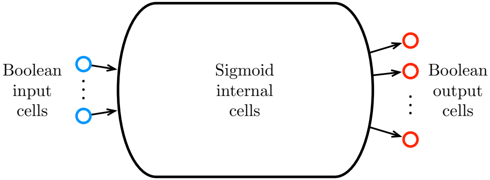
$$\exists F \subseteq (\Sigma \times \{0,1\})^\omega \text{ such that } F \text{ is } \Pi_2^0 \text{ and}$$

$$E \text{ is the projection of } F \text{ onto } \Sigma^\omega$$

A set $E \subseteq \Sigma^\omega$ is in the class Π_1^1 iff $\Sigma^\omega - E$ is in Σ_1^1.
Suslin's Theorem states that : Borel sets $= \Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$

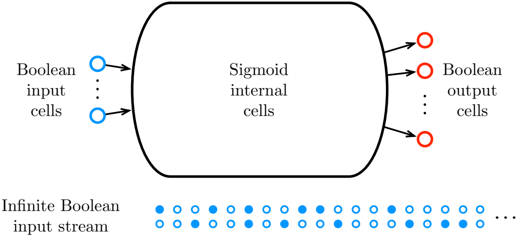
Deterministic ω-NNs

We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



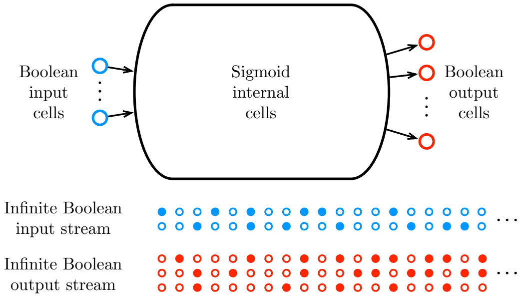
Deterministic ω-NNs

We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



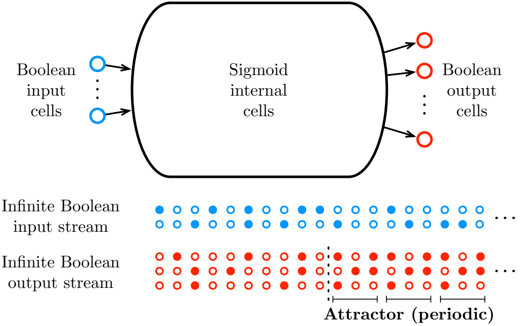
Deterministic ω-NNs

We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



Deterministic ω-NNs

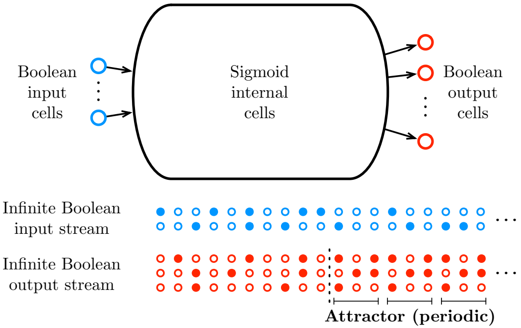
We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



- The attractors are assumed to be classified into two possible kinds: *accepting* or *rejecting*.

Deterministic ω-NNs

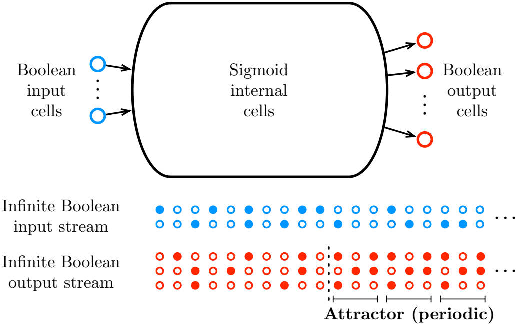
We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



- An infinite Boolean input stream is *accepted* by $\mathcal{N}$ if the corresponding Boolean output stream visits a *accepting* attractor.

Deterministic ω-NNs

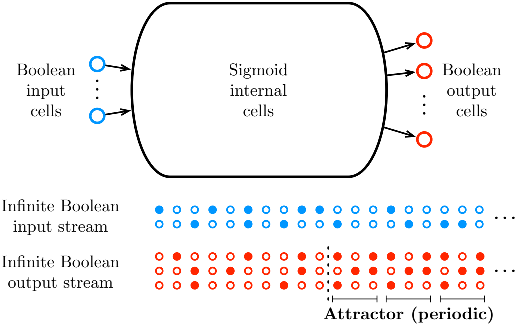
We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



- An infinite Boolean input stream is *rejected* by $\mathcal{N}$ if the corresponding Boolean output stream visits a *rejecting* attractor.

Deterministic ω-NNs

We consider RNNs with Boolean input and output cells, sigmoidal internal cells, and working on infinite input streams.



- The set of all input streams that are accepted by $\mathcal{N}$ is the ω-*language* recognized by $\mathcal{N}$.

Deterministic ω-NNs

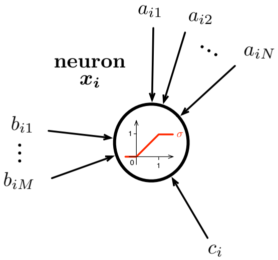
We consider two models of deterministic NNs:

1. static rational NNs: D-St-NN[$\mathbb{Q}$]s
2. bi-valued evolving rational NNs: D-Ev$_2$-NN[$\mathbb{Q}$]s

Deterministic ω-NNs

We consider two models of deterministic NNs:

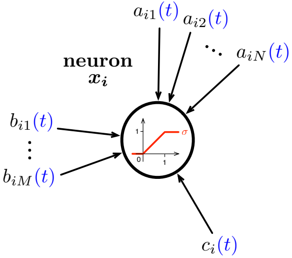
1. static rational NNs: D-St-NN[$\mathbb{Q}$]s
2. bi-valued evolving rational NNs: D-Ev$_2$-NN[$\mathbb{Q}$]s



Deterministic ω-NNs

We consider two models of deterministic NNs:

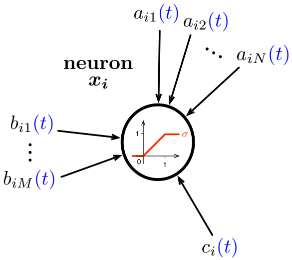
1. static rational NNs: D-St-NN[$\mathbb{Q}$]s
2. bi-valued evolving rational NNs: D-Ev$_2$-NN[$\mathbb{Q}$]s



Deterministic ω-NNs

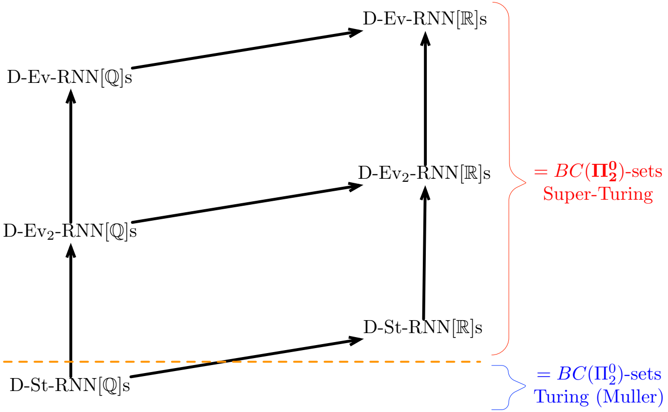
We consider two models of deterministic NNs:

1. static rational NNs: D-St-NN[$\mathbb{Q}$]s
2. bi-valued evolving rational NNs: D-Ev$_2$-NN[$\mathbb{Q}$]s



Results

Relationship between the models:



Results

Theorem (Staiger (1997), Cabessa & Villa (2016))

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in BC(\Pi^0_2)$
- *L is recognizable by some deterministic Muller TM*
- *L is recognizable by some D-St-NN[$\mathbb{Q}$]*

Results

Theorem (Cabessa & Villa (2016))

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in BC(\mathbf{\Pi^0_2})$;
- L *is recognizable by some D-Ev*$_2$*-NN*$[\mathbb{Q}]$;

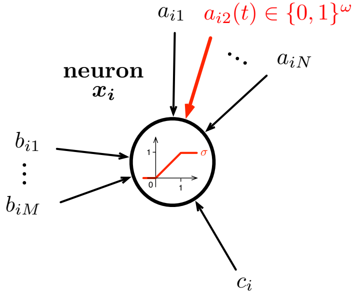
Results – Summary

DET.	Static	Bi-valued Evolving
	D-St-NN[$\mathbb{Q}$]s	D-Ev$_2$-NN[$\mathbb{Q}$]s
	$= BC(\Pi_2^0)$	$= BC(\mathbf{\Pi_2^0})$
	Turing (Muller)	super-Turing

Deterministic ω-NNs

- We consider bi-valued evolving rational NNs with only one evolving weight:

$$\text{D-Ev}_2\text{-NN}[\mathbb{Q}, \alpha]\text{s, where } \alpha \in \{0,1\}^\omega$$



Relativized Classes

If $L \subseteq X^\omega$ and Γ is a topological class, then for $\alpha \in \{0,1\}^\omega$,

$L \in \Gamma(\alpha)$ iff

there exists some $L' \subseteq (X \times \{0,1\})^\omega$ such that $L' \in \Gamma$, and

$$[(\forall x \in X^\omega) \;\; x \in L \Longleftrightarrow (x, \alpha) \in L']$$

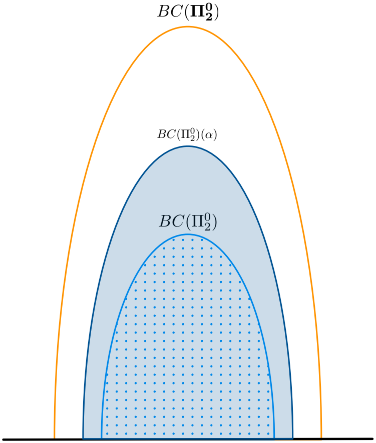
i.e. $L \in \Gamma(\alpha)$ iff L is the section of $L' \in \Gamma$ in α

Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in BC(\Pi^0_2)(\alpha)$, *for some* $\alpha \in \{0,1\}^\omega$
- L *is recognizable by some D-Ev*$_2$*-NN*$[\mathbb{Q}, \alpha]$



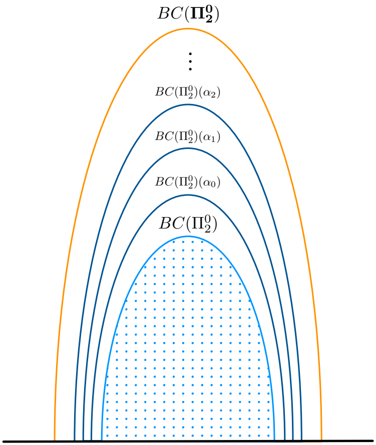
Results

Proposition

There exist some infinite sequence $(\alpha_k)_{k<\omega_1}$, where each $\alpha_i \in \{0,1\}^\omega$, such that

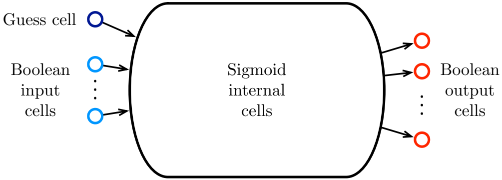
$$BC(\Pi^0_2)(\alpha_i) \subsetneq BC(\Pi^0_2)(\alpha_j)$$

for all $i < j < \omega_1$.



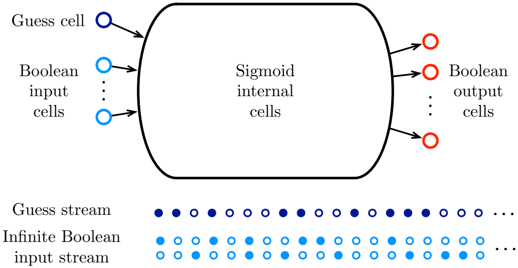
Nondeterministic ω-NNs

The NNs are provided with an additional Boolean guess cell.



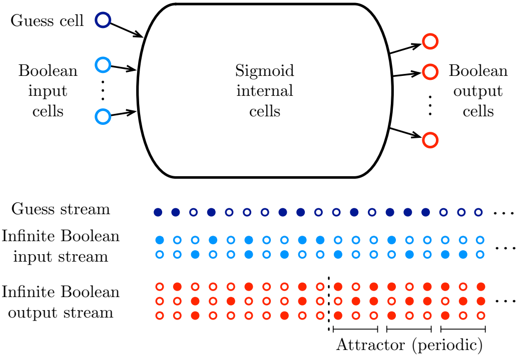
Nondeterministic ω-NNs

The NNs are provided with an additional Boolean guess cell.



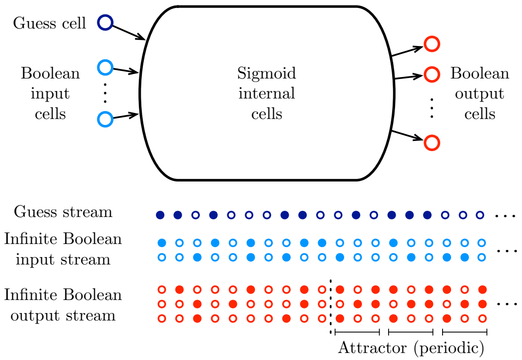
Nondeterministic ω-NNs

The NNs are provided with an additional Boolean guess cell.



Nondeterministic ω-NNs

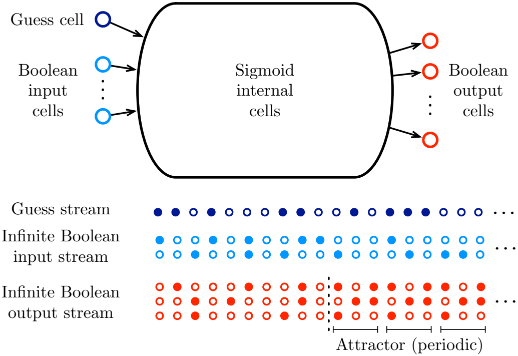
The NNs are provided with an additional Boolean guess cell.



- The attractors are assumed to be classified into two possible kinds: *accepting* or *rejecting*.

Nondeterministic ω-NNs

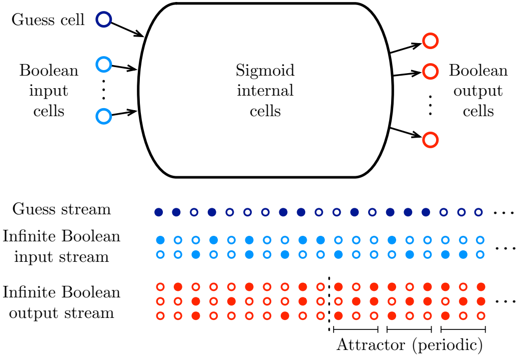
The NNs are provided with an additional Boolean guess cell.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *accepted* by $\mathcal{N}$ iff there exists some guess $g \in \mathbb{B}^\omega$ s.t. $\mathcal{N}(s, g)$ enters an accepting attractor.

Nondeterministic ω-NNs

The NNs are provided with an additional Boolean guess cell.



- Input stream $s \in (\mathbb{B}^M)^\omega$ *rejected* by $\mathcal{N}$ iff for all guess $g \in \mathbb{B}^\omega$, $\mathcal{N}(s, g)$ does not enter an accepting attractor.

Nondeterministic ω-NNs

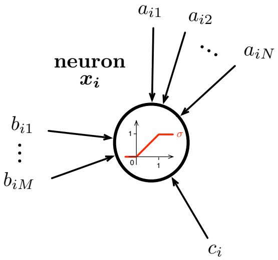
We consider two models of nondeterministic NNs:

1. static rational NNs: N-St-NN[$\mathbb{Q}$]s
2. bi-valued evolving rational NNs: N-Ev$_2$-NN[$\mathbb{Q}$]s

Nondeterministic ω-NNs

We consider two models of nondeterministic NNs:

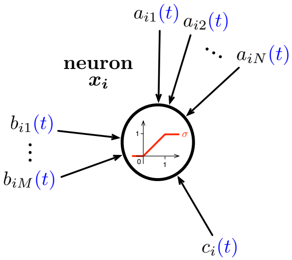
1. static rational NNs: N-St-NN[$\mathbb{Q}$]s
2. bi-valued evolving rational NNs: N-Ev$_2$-NN[$\mathbb{Q}$]s



Nondeterministic ω-NNs

We consider two models of nondeterministic NNs:

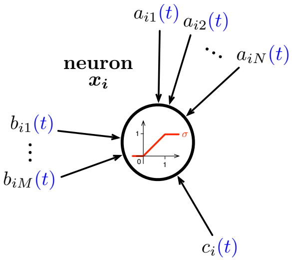
1. static rational NNs: N-St-NN[$\mathbb{Q}$]s
2. bi-valued evolving rational NNs: N-Ev$_2$-NN[$\mathbb{Q}$]s



Nondeterministic ω-NNs

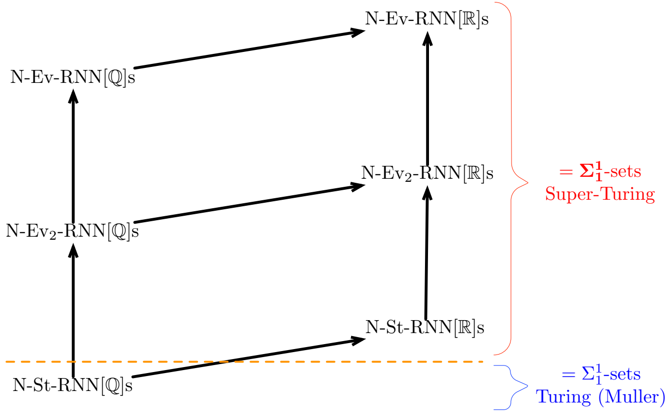
We consider two models of nondeterministic NNs:

1. static rational NNs: N-St-NN[$\mathbb{Q}$]s
2. bi-valued evolving rational NNs: N-Ev$_2$-NN[$\mathbb{Q}$]s



Results

Relationship between the models:



Results

Theorem (Staiger (1997), Cabessa & Villa (2016))

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in \Sigma^1_1$
- L *is recognizable by some nondeterministic Muller TM*
- L *is recognizable by some N-St-NN[$\mathbb{Q}$]*

Results

Theorem (Cabessa & Villa (2016))

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in \mathbf{\Sigma}^1_1$;
- L *is recognizable by some N-Ev$_2$-NN[$\mathbb{Q}$];*

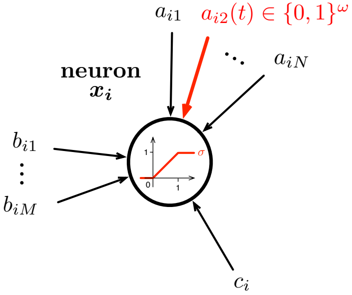
Results – Summary

NONDET.	Static	Bi-valued Evolving
	N-St-NN[$\mathbb{Q}$]s	N-Ev$_2$-NN[$\mathbb{Q}$]s
	$= \Sigma_1^1$	$= \boldsymbol{\Sigma}_1^1$
	Turing (Muller)	super-Turing

Nondeterministic ω-RNNs

1. We consider bi-valued evolving rational NNs with only one evolving weight:

$$\text{N-Ev}_2\text{-NN}[\mathbb{Q}, \alpha]\text{s, where } \alpha \in \{0,1\}^\omega$$

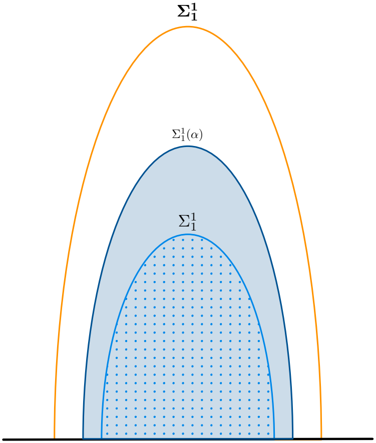


Results

Theorem

Let $L \subseteq (\mathbb{B}^M)^\omega$. The following conditions are equivalent.

- $L \in \Sigma^1_1(\alpha)$, *for some* $\alpha \in \{0,1\}^\omega$
- L *is recognizable by some N-Ev$_2$-NN[$\mathbb{Q}, \alpha$]*



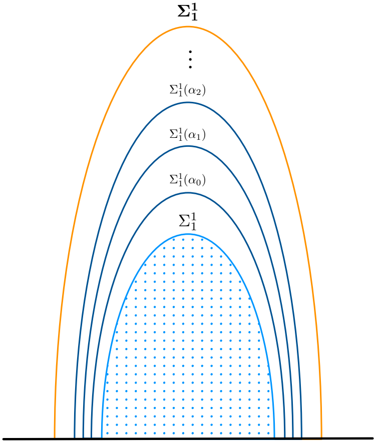
Results

Proposition

There exist some infinite sequence $(\alpha_k)_{k<\omega_1}$, *where each* $\alpha_i \in \{0,1\}^\omega$, *such that*

$$\Sigma^1_1(\alpha_i) \subsetneq \Sigma^1_1(\alpha_j)$$

for all $i < j < \omega_1$.



Conclusion

- We provided a precise characterization of the expressive power of evolving neural networks employing only one evolving weight.
- As a consequence, a proper hierarchy of classes of evolving neural nets, based on the complexity of their underlying evolving weights, can be obtained.
- The hierarchy contains chains of length ω_1 as well as uncountable antichains.
- The super-Turing computational capabilities of neural models is related to the issue of *hypercomputation*.
- Current physical theories are consistent with the possibility of hypercomputational systems (quantum, relativistic, etc.). No such systems are currently feasible or harnessable.