

# Contextuality in Multipartite Pseudo-Telepathic Graph Games

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# Introduction

**Contextuality / non locality: very foundation of quantum mechanics.**

- Quantum mechanics is contextual [Kochen–Specker67]
- Active area of research: understand the mathematical structures of contextuality [Abramsky et al.; Acín et al.]

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**Pseudo-telepathic games**

- Collaborative games: all the players win or all the players lose.
- No communication between the players.
- Shared resources (random bits, entangled states...)
- Graph games.
- Interesting cases: the games that can be won quantumly but not classically (pseudo-telepathic game).

## Multipartite Games

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- Question:  $x \in \{0, 1\}^V$ : each player  $u \in V$  receives a single bit  $x_u \in \{0, 1\}$  of the question.
- Answer:  $a \in \{0, 1\}^V$ : each player  $u \in V$  produces a single bit  $a_u \in \{0, 1\}$  of the answer, without communication.
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E.g., when  $V = \{1, 2, 3\}$ ,

question $x \in \{0, 1\}^V$	$a \in \{0, 1\}^V$ should satisfy
100	$2 a  \neq 2 \pmod{4}$ $\Leftrightarrow  a  \neq 1 \pmod{2}$ $\Leftrightarrow  a  = 0 \pmod{2}$
010	$ a  = 0 \pmod{2}$
001	$ a  = 0 \pmod{2}$
111	$ a  = 1 \pmod{2}$

## No Classical Winning Strategy for the Mermin Game

**Property.** There is no winning strategy for the Mermin game when  $|V| \geq 3$ .

**Proof.** For any  $u \in V$ , let  $f_u : \{0, 1\} \rightarrow \{0, 1\}$  be the deterministic strategy of player  $u$ :  $a_u = f_u(x_u)$ .

$$\begin{array}{rcccccc} x = 100 & f_1(1) & + & f_2(0) & + & f_3(0) & = & 0 \text{ mod } 2 \\ x = 010 & f_1(0) & + & f_2(1) & + & f_3(0) & = & 0 \text{ mod } 2 \\ x = 001 & f_1(0) & + & f_2(0) & + & f_3(1) & = & 0 \text{ mod } 2 \\ x = 111 & f_1(1) & + & f_2(1) & + & f_3(1) & = & 1 \text{ mod } 2 \end{array}$$

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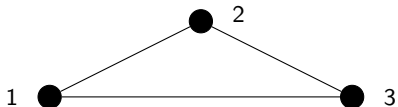
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Assume the players share a graph state  $|G_\Delta\rangle$ .

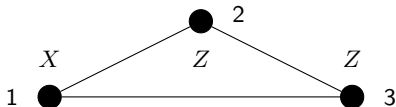


**Quantum strategy:**

- If  $x_u = 0$ ,  $a_u$  is the outcome of the measure according to  $Z$  of qubit  $u$ ;
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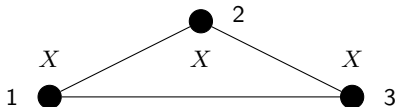
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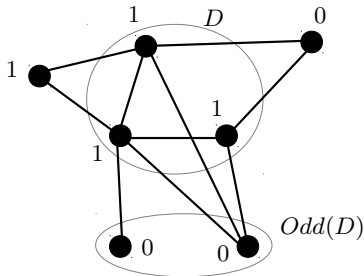
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 $\Leftrightarrow X_1 Z_2 X_2 Z_2 X_3 |G_\Delta\rangle = |G_\Delta\rangle$  since  $Z^2 = I$   
 $\Leftrightarrow X_1 X_2 X_3 |G_\Delta\rangle = -|G_\Delta\rangle$  since  $XZ = -ZX$ .  
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# Graph Games

**Definition. [Anshu, Mhalla'13]** Given a graph  $G = (V, E)$ , let  $\mathcal{L}_G := \{(a|x) : \exists D \text{ involved in } x \text{ s.t. } \sum_{u \in D \cup \text{Odd}(D)} a_u = |G[D]| + 1 \pmod{2}\}$ , where

- $\text{Odd}(D) = \{u \in V \mid |N(u) \cap D| = 1 \pmod{2}\}$  is the odd neighbourhood of  $D$ ;
- $|G[D]|$  is the number of edges of the subgraph induced by  $D$ ;
- $D$  is involved with  $x$  if  $\forall u \in D, x_u = 1$  and  $\forall u \in \text{Odd}(D), x_u = 0$ .



Losing condition:

$$\begin{aligned} \sum_{u \in D \cup \text{Odd}(D)} a_u &= |G[D]| + 1 \pmod{2} \\ &= 1 \pmod{2} \end{aligned}$$

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**QStrat:** Given a graph  $G = (V, E)$ , the players share the graph state  $|G\rangle$ ,

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**Proof.** Given a  $x \in \{0, 1\}^V$  and  $D \subseteq V$  involved with  $x$ ,

$$\prod_{u \in D} X_u Z_{N(u)} |G\rangle = |G\rangle \Leftrightarrow X_D Z_{\text{Odd}(D)} |G\rangle = (-1)^{|G[D]|} |G\rangle$$

Since  $D$  is involved, qubits in  $D$  are  $X$ -measured, those in  $\text{Odd}(D)$  are  $Z$ -measured, so

$$\sum_{u \in D \cup \text{Odd}(D)} a_u = |G[D]| \pmod{2}$$

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**Property. QStrat** produces the good answers **uniformly**:

$$p(a|x) = \begin{cases} 0 & \text{if } (a|x) \in \mathcal{L} \\ \frac{|\{D \text{ involved in } x\}|}{2^{|V|}} & \text{otherwise.} \end{cases}$$

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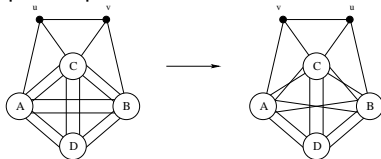
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**Property.** When the players share a graph state  $|G\rangle$ , they can win any game described by a graph pivot-equivalent to  $G$ .



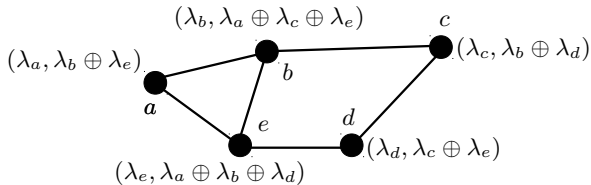


## Classical Strategy

**CStrat:** Given a graph  $G = (V, E)$ , pick uniformly at random  $\lambda \in \{0, 1\}^V$ .

Each player  $u \in V$  receives  $(\lambda_u, \mu_u)$ , where  $\mu_u = \sum_{v \in N_G(u)} \lambda_v \bmod 2$ .

Given a question  $x \in \{0, 1\}^V$ , each player  $u \in V$  locally computes and answers  $a_u = (1 - x_u) \cdot \lambda_u + x_u \cdot \mu_u \bmod 2$ .



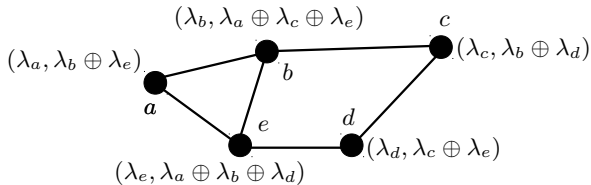
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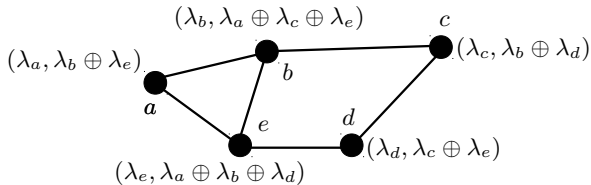
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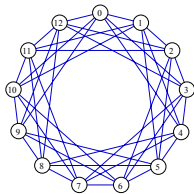
**Theorem [Anshu, Mhalla'13]** If  $G$  is not bipartite, there is no winning classical strategy.

**Property.** For any graph  $G$ , **CStrat** can be used to turn any winning strategy into a uniform winning strategy.

# Multipartite width, a measure of contextuality

- Bipartite graph games have multipartite width 1.
- Mermin games ( $K_n$ ) have multipartite width 2.
- The game associated with a Paley graph  $\mathcal{P}_n$  has multipartite width at least  $\log_2(n)$  [Anshu, Mhalla'13]

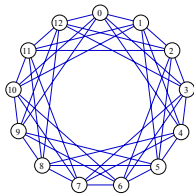
Paley graph  $\mathcal{P}_n$ ,  $n = 1 \pmod 4$  prime,  
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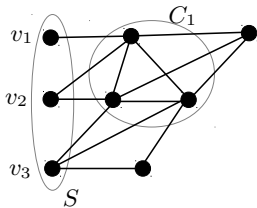
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**Theorem.** For any  $n > n_0$ , there exist graph games on  $n$  players of multipartite width at least  $\lfloor 0.11n \rfloor$ .

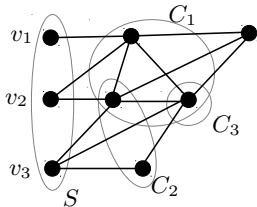
## Linear multipartite width

**Lemma.** [Anshu, Mhalla'13] A graph game associated with  $G$  has multipartite width at least  $k$  if  $G$  is  $k$ -odd dominated, i.e. if for any  $S \in \binom{V}{k}$ , there exists a labelling of the vertices in  $S = \{v_1, \dots, v_k\}$  and  $C_1, \dots, C_k$ , s.t.  $\forall i, C_i \subseteq V \setminus S$  and  $\text{Odd}(C_i) \cap \{v_1, \dots, v_k\} = \{v_i\}$  and  $C_i \subseteq V \setminus \text{Odd}(C_i)$ .



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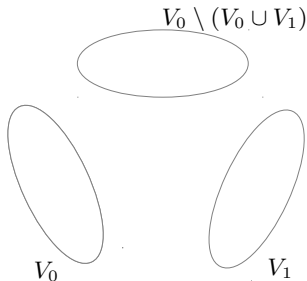
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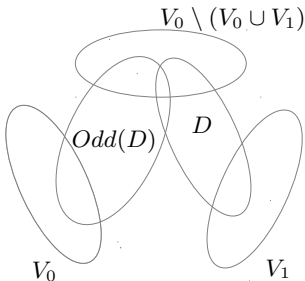




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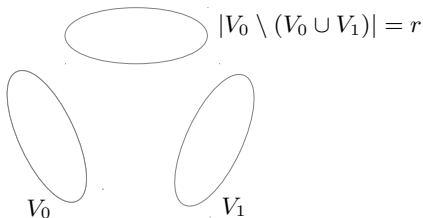


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**Theorem.** For any even  $n > n_0$ , there exists a non-bipartite  $\lfloor 0.110n \rfloor$ -odd dominated graph of order  $n$ .



## Conclusion & Perspectives

- For all graph  $G$  there exists a quantum strategy to win the corresponding graph game.
- Uniformisation of winning strategies.
- Invariance under pivot transformation
- Existence of graph games with a linear multipartite width.

### Perspectives

- Improve the bound for the Paley graphs.
- Upper bound on multipartite width?
- Algorithm for multipartite width.